Mean square stabilization over SNR-constrained channels with colored and mutually correlated additive noises

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***Abstract*— This paper addresses the problem of stabilizing a MIMO discrete-time LTI system over a MIMO additive noise channel. We assume that the communication link between the plant output and the controller input consists of multiple additive colored mutually correlated noise channels subject to independent signal-to-noise ratio (SNR) constraints. We derive analytical conditions for which mean square stabilization (MSS) can be achieved under such constraints. We also formulate numerical methods to test these conditions when the noise is white and correlated. Moreover, for simpler plant models, a characterization of the set of power constraints compatible with MSS is obtained. Our results show that the frequency response of the spectral factor related to the channel noise affects the minimum SNR for stability depending mostly on the unstable poles and their directions. This is aggravated by the existence of non minimum phase zeros and higher relative degree of the plant. On the other hand, we detect that high correlated noise show lower SNR requirements for stability compared to independent noise channels. Numerical simulations are provided to illustrate the theoretical results.**

***Index Terms*— Networked control systems, colored noise channel, mean-square stabilization, signal-to-noise ratio constraints.**

I. Introduction

The analysis of feedback control systems under communication constraints, known alternatively as Networked Control Systems (NCSs) has been a topic of active research in the recent years (see [1], [2], and the references therein). A natural question in studying NCSs is whether there exist fundamental limits for feedback stabilization over limited communication channels. Some of the works on this matter involve communication channels with signal-to-noise ratio (SNR) and capacity constraints [3], [4], quantization noises [5], multiplicative noises [6], data loss [7] and data-rate limits [8]. In this paper we focus on additive noises channels with SNR constraints.

Stabilizability over SNR constrained channels was studied for single-input single-output (SISO) plants in [3], [9], where it was shown that the minimun SNR for stabilizability is determined by the plant's relative degree, non-minimum phase (NMP) zeros and unstable poles. Extensions of these results to MIMO plants have been presented under different assumptions. In [10], [11], additive white noise channels were considered in a two-parameter controller configuration with the assumption that the total channel input power, i.e., the sum of the input power of individual channels, is constrained. On the other hand, in [12]–[14], constraints were also imposed on the power of each SISO channel separately. These contributions also show that stabilization of MIMO plants under individual channel constraints depends on the NMP zeros and unstable poles, in both their locations and directions.

The previous research alluded to above was conducted with the assumption that the additive noise is white. However, with more general channel noises, more realistic models of communication links can be analyzed. Previous works dealing with colored additive noise include [15], [16], where a tight condition for stability of SISO plants under SNR constraints in a colored channel architecture can be found. For MIMO plants, the channel noise vector components could be not only colored but also mutually correlated. In the literature, few papers dealing with such setup can be found (see e.g. [17], [18]), and in those papers the SNR-constrained stabilization problem is not considered.

In this paper, we present analytical conditions for which a MIMO discrete-time linear time-invariant plant can be stabilized over a MIMO channel which contains additive colored and correlated noises and is subject to multiple SNR constraints. This setup is a generalization of the one considered in [19], where we present preliminary results of the current paper, and where only a correlated but white noise vector is considered. Our main purpose in this paper is to examine how colored noises may correlate across different channels to further degrade a system's stabilizability. This proceeds as follows. First, we obtain the set of conditions for output feedback stabilization for minimum phase systems. Subsequently, we prove that these conditions can be reformulated equivalently as a linear matrix inequality (LMI) and as a modified algebraic Riccati equation (MARE) with a matrix inequality when the channel contains mutually correlated and white noises. We also obtain more general conditions valid for non-minimum phase systems. Simulation examples are provided in both cases, to illustrate the complex relationship between the noise spectra, the plant's unstable poles and non-minimum zeros, and the mean square stability satisfying SNR constraints.

The remainder of this article is as follows. Section II states the main problem to be studied in this work and introduces some necessary preliminaries. In Section III we study stabilization under SNR constraints for minimum phase systems with additive colored and correlated noise channels. Section IV examines non-minimum phase plants. Numerical simulations are given in Section V, while conclusions are drawn in Section VI.

Notation: For any complex number z, we denote its complex conjugate as z^*. For any vector v, we denote $\|v\|$ as its Euclidean norm. Given any matrix M, its hermitian is denoted by M^{H} and its transpose by $M^\top$. The symbol $\odot$ denotes the Hadamard product. We use $\text{diag}(M_1, \ldots, M_n)$ to denote a (block) diagonal matrix with (block) diagonal elements $M_1, \ldots, M_n$. Denote by $\mathcal{R}$ the set of all real rational discrete-time transfer functions. We define the following subsets of $\mathcal{R}$: $\mathcal{RH}_\infty$ contains all stable and proper transfer functions, $\mathcal{RH}_2$ contains all stable and strictly proper transfer functions and $\mathcal{RH}_2^\perp$ contains all the transfer functions that have no poles inside or on the unit circle. The Hermitian of a transfer matrix $H(z)$ is denoted by $H(z)^\sim$. If $H(z)$ is a transfer function with no poles on the unit circle, then the 2-norm of $H(z)$ is denoted by $\|H(z)\|_2$. For simplicity, the dependence on z is sometimes omitted.

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II. Problem formulation and preliminaries

We consider the control architecture depicted in Fig. 1 a), where $G(z)$ is a MIMO discrete-time LTI plant to be controlled by a proper discrete-time controller $K(z)$. The control signal is denoted by $u \in \mathbb{R}^{n_u}$. The MIMO channel is described by $\tilde{y} = y + q$, where $\tilde{y} \in \mathbb{R}^{n_y}$ is the channel output received at the controller, $y \in \mathbb{R}^{n_y}$ is the plant output, and $q \in \mathbb{R}^{n_y}$ is a vector of stochastic additive noises, which is assumed to be a zero-mean wide sense stationary (WSS) process with spectrum $S_q(z)$.

Definition 2.1: The noise q is said to be white if its spectrum $S_q(z) = P_q$, where P_q is the stationary covariance matrix of q (and hence constant); otherwise is said to be colored. On the other hand, the noise q is said to be uncorrelated if its spectrum $S_q(z)$ is a diagonal matrix; otherwise is said to be correlated.

Throughout this paper we will make the following assumptions.

Assumption 2.1:

- $G(z)$ is a strictly proper, full row normal rank LTI system.
- $G(z)$ has no poles on the unit circle. Moreover, all the unstable poles of $G(z)$ are simple (with algebraic multiplicity one).
- $S_q(z)$ does not have zeros on the unit circle. It admits the spectral factorization $S_q(z) = \Omega_q(z)\Omega_q(z)^{\sim}$, with $\Omega_q(z) \in \mathcal{RH}_\infty$, and $\Omega_q^{-1}(z) \in \mathcal{RH}_\infty$.
- The initial state of the plant is a second order random variable, uncorrelated with q.

Note that q is in general a vector of mutually correlated and colored additive noises. We also assume that each channel component pair (y_i, q_i) is subject to a stationary SNR constraint given by

$$\gamma_i := \frac{\sigma_{y_i}^2}{\sigma_{q_i}^2} < \Gamma_i, \quad i = 1, 2, \ldots, n_y, \tag{1}$$

where $\sigma_{y_i}^2$ and $\sigma_{q_i}^2$ are the stationary variances of the i-th component of y and q respectively, γ_i is the SNR on the i-th channel and $\Gamma_i \in \mathbb{R}^+$ is its upper limit.

Our goal in this paper is to find necessary and sufficient conditions such that under the above assumptions the NCS in Fig. 1.a) can be stabilized in spite of the multiple SNR constraints in (1). Due to the random nature of the channel, we will use *mean square stability* (MSS) as the notion of stability.

Definition 2.2: The NCS in Fig. 1.a) is said to be mean square stable (MSS) if the state covariance matrix of the NCS converges asymptotically to a finite matrix regardless of the initial state.

The mean square stabilizability problem stated above can be alternatively formulated as an optimal design problem. To this end, we can model the noise q as in the architecture depicted in Fig.1.b), where $\hat{q} \in \mathbb{R}^{n_y}$ is a zero-mean WSS white noise process with identity covariance matrix. So, it is possible to express y in terms of $\hat{q}$ as

$$\begin{aligned} y &= (I - G(z)K(z))^{-1}G(z)K(z)\Omega_q(z)\hat{q} \\ &=: T_{qy}(z)\Omega_q(z)\hat{q}. \end{aligned}$$

It is well-known that for an LTI system with a second order initial state and second order WSS inputs, a controller that achieves internal stability also achieves MSS [20]. Hence, we can use the Youla parametrization to describe the set of all mean square stabilizing controllers. For this purpose, we consider a doubly coprime factorization of G over $\mathcal{RH}_\infty$ such that $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$, where $N, M, \tilde{N}, \tilde{M} \in \mathcal{RH}_\infty$, and

$$\begin{bmatrix} \tilde{X}(z) & -\tilde{Y}(z) \\ -\tilde{N}(z) & \tilde{M}(z) \end{bmatrix} \begin{bmatrix} M(z) & Y(z) \\ N(z) & X(z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

holds for some $\tilde{X}, \tilde{Y}, X$ and Y in $\mathcal{RH}_\infty$. It is also well-known that any LTI proper stabilizing controller K can be written in terms of the double coprime factorization of G as $K = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M})$,

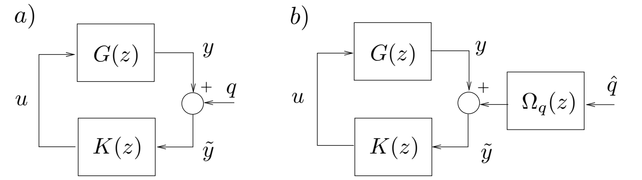

Fig. 1: Networked control system under analysis. a) Original setup. b) Alternative setup with noise q modeled by $\Omega_q(z)$.

where Q is in $\mathcal{RH}_\infty$. Using this parametrization, the transfer function T_{qy} can be written as

$$T_{qy}(z) = N(z)\tilde{Y}(z) - N(z)Q(z)\tilde{M}(z). \tag{2}$$

To enforce the SNR constraints on each channel, we first calculate the variance of $y_i, i \in \{1, 2, \ldots, n_y\}$, as $\sigma_{y_i}^2 = \|\eta_i^\top T_{qy}(z)\Omega_q(z)\|_2^2$, where η_i denotes the i-th column of the $n_y \times n_y$ identity matrix. We also know that the variance of the i-th channel noise is given by $\sigma_{q_i}^2 = \|\eta_i^\top \Omega_q(z)\|_2^2$. Thus, for the system depicted in Fig. 1 b) to achieve MSS and satisfy the SNR conditions, it is both necessary and sufficient that

$$\max_{i\in\{1,2,\ldots,n_y\}} \min_{Q\in\mathcal{RH}_\infty} \frac{\|\eta_i^\top T_{qy}(z)\Omega_q(z)\|_2^2}{\Gamma_i\|\eta_i^\top \Omega_q(z)\|_2^2} < 1. \tag{3}$$

The condition expressed in (3) permits us to obtain the minimum SNR that each channel must suffice in order to stabilize the system in Fig. 1 b), which is the same condition to stabilize the system in Fig. 1 a). It is clear that the conditions depend strongly on the spectral factor $\Omega_q(z)$. Hence, understanding the effect of the noise's color and/or correlation is essential for the stabilization analysis of this type of systems.

Denote by $\mathcal{P} \triangleq \{p_1, \ldots, p_{n_p}\}$ the set of unstable poles of $G(z)$. We introduce the factorization

$$\tilde{M}(z)\Omega_q(z) = \tilde{M}_m(z)R(z), \tag{4}$$

where $\tilde{M}_m(z) \in \mathcal{RH}_\infty$ is biproper and minimum-phase, and $R(z)$ is an all-pass filter given by

$$\begin{aligned} R(z) &= R_{n_p}(z)R_{n_p-1}(z)\cdots R_1(z), \\ R_k(z) &= \left(\frac{z - p_k}{1 - p_k^* z} - 1\right)\nu_k\nu_k^{\mathsf{H}} + I, \end{aligned}$$

and ν_k is the unique unitary vector satisfying

$$\tilde{M}(p_k)\Omega_q(p_k)R_0^{-1}(p_k)R_1^{-1}(p_k)\cdots R_{k-1}^{-1}(p_k)\nu_k = 0,$$

with $R_0(z) = I$.

We also must introduce the following technical lemma.

Lemma 2.1 ([13]): Consider transfer function matrices $\mathcal{U} \in \mathcal{RH}_\infty^{a\times b}$, $\mathcal{V} \in \mathcal{RH}_\infty^{a\times p}$, and $\mathcal{W}_i \in \mathcal{RH}_\infty^{q\times b}, i \in \{1, \ldots, a\}$ with $a, b, p, q \in \mathbb{N}$. Assume that, for every $i \in \{1, \ldots, a\}$, $\mathcal{V}$ and $\mathcal{W}_i$ have no zeros on the unit circle, $\mathcal{V}$ is right invertible, and $\mathcal{W}_i$ is left invertible. Denote by η_i the i-th column of the $a \times a$ identity matrix. Consider, for $i \in \{1, 2, \ldots, a\}$, the minimization problems

$$\begin{aligned} J_{\inf_i} &\triangleq \inf_{Q\in\mathcal{RH}_\infty} \|\eta_i^\top \mathcal{U} + \eta_i^\top \mathcal{V}Q\mathcal{W}_i\|_2^2, \\ Q_i^{\text{opt}} &\triangleq \arg\inf_{Q\in\mathcal{RH}_\infty} \|\eta_i^\top \mathcal{U} + \eta_i^\top \mathcal{V}Q\mathcal{W}_i\|_2^2. \end{aligned}$$

If the NMP zeros of $\mathcal{V}$, including zeros at infinity, have only canonical input directions, or $\mathcal{V}$ has no NMP zeros, then there exists $Q^{\text{opt}} \in \mathcal{RH}_\infty$ such that $J_{\inf_i} = \|\eta_i^\top \mathcal{U} + \eta_i^\top \mathcal{V}Q^{\text{opt}}\mathcal{W}_i\|_2^2$, where $Q^{\text{opt}} = \sum_{i=1}^a Q_i^{\text{opt}}$, for some $Q_i^{\text{opt}} \in \mathcal{RH}_\infty$.

Proof: See details in [13]. ■

Lemma 2.1 states that, under certain assumptions, it is possible to solve a set of simultaneous minimization problems using one unique optimal Youla parameter.

III. Stabilizability of minimum-phase plants

Conductive to showing the basic effects of color and correlation in this stabilization problem, we obtain an explicit stabilization condition for plants with no finite NMP zeros, and with a relative degree 1. The result is stated in Theorem 3.1.

Theorem 3.1: Consider the NCSs depicted in Fig. 1. Assume that Assumption 2.1 holds, $G(z)$ has a relative degree of 1 with distinct unstable poles $\{p_i\}_{i=1}^{n_p}$, and $G(z)$ has no finite NMP zeros. Then, the feedback loops in Fig. 1 are MSS and the SNR constraints in (1) are satisfied if and only if

$$\max_{i\in\{1,2,\dots,n_y\}} \frac{1}{\sigma_{q_i}^2\Gamma_i}\eta_i^\top\left(\sum_{\ell=1}^{n_p}\sum_{k=1}^{n_p}\frac{\mathcal{A}_\ell\mathcal{A}_k^{\mathsf{H}}}{p_\ell p_k^*-1}\right)\eta_i<1, \tag{5}$$

where

$$\mathcal{A}_\ell \triangleq \Omega_q(p_\ell)\left[(z-p_\ell)R^{-1}(z)\right]\Big|_{z=p_\ell}. \tag{6}$$

Proof: Using the parametrization in (2), the factorization in (4), Lemma 1 from [14], and standard 2-norm properties, we can express the argument of the minimization problem in (3) as follows:

$$\begin{aligned}J_i &:= \frac{1}{\sigma_{q_i}^2\Gamma_i}\|\eta_i^\top(N\tilde{Y}\Omega_q - NQ\tilde{M}\Omega_q)\|_2^2\\ &= \frac{1}{\sigma_{q_i}^2\Gamma_i}\|\eta_i^\top(N\tilde{Y}\Omega_q - NQ_i\tilde{M}\Omega_q)\|_2^2\\ &= \frac{1}{\sigma_{q_i}^2\Gamma_i}\|\eta_i^\top(X\tilde{M}\Omega_q - \Omega_q - NQ_i\tilde{M}\Omega_q)\|_2^2\\ &= \frac{1}{\sigma_{q_i}^2\Gamma_i}\|\eta_i^\top(X\tilde{M}_m - \Omega_q R^{-1} - NQ_i\tilde{M}_m)\|_2^2.\end{aligned}$$

Decompose $\Omega_q(z)R^{-1}(z)$ by an orthogonal projection as

$$\Omega_q(z)R^{-1}(z) = \Delta(z)+\nabla(z). \tag{7}$$

with $\Delta(z)\in\mathcal{RH}_2$ and $\nabla(z)\in\mathcal{RH}_2^\perp$. It follows by the orthogonality property of the 2-norm that

$$\begin{aligned}J_i = \frac{1}{\sigma_{q_i}^2\Gamma_i}\Big(&\left\|\eta_i^\top(\nabla(\infty)-\nabla(z))\right\|_2^2\\ &+\left\|\eta_i^\top(X\tilde{M}_m-\nabla(\infty)-\Delta-NQ_i\tilde{M}_m)\right\|_2^2\Big).\end{aligned}$$

Since all the unstable poles $\{p_\ell\}_{\ell=1}^{n_p}$ of $G(z)$ are simple, $\nabla(z)$ can be expressed as

$$\nabla(z) = \Omega_q(\infty)R^{-1}(\infty)+\sum_{\ell=1}^{n_p}\frac{\mathcal{A}_\ell}{z-p_\ell}, \tag{8}$$

with $\mathcal{A}_\ell$ determined as in (6). With (8), we can write the minimum of J_i under the optimal Youla parameter Q_i^{opt} as

$$\begin{aligned}J_i(Q_i^{\text{opt}}) &= \frac{1}{\sigma_{q_i}^2\Gamma_i}\|\eta_i^\top(\Omega_q(\infty)R^{-1}(\infty)-\nabla(z))\|_2^2\\ &= \frac{1}{\sigma_{q_i}^2\Gamma_i}\left\|\eta_i^\top\left(\sum_{\ell=1}^{n_p}\frac{\mathcal{A}_\ell}{z-p_\ell}\right)\right\|_2^2 \\ &= \frac{1}{\sigma_{q_i}^2\Gamma_i}\sum_{\ell=1}^{n_p}\sum_{k=1}^{n_p}\frac{\eta_i^\top\mathcal{A}_\ell\mathcal{A}_k^{\mathsf{H}}\eta_i}{p_\ell p_k^*-1}.\end{aligned} \tag{9}$$

Our claim follows by (3). ■

Theorem 3.1 shows that system stabilization under the SNR constraints depends on the noise characteristics represented by the spectral factorization matrix $\Omega_q(z)$. It particular, it is related to the evaluation of $\Omega_q(z)$ at the unstable poles of the plant. Hence, the relationship between the unstable poles of $G(z)$ and the zeros of $\Omega_q(z)$ can play an important role. This observation coincides with [21] for SISO continuous-time plants. Also, note that the pole output directions are present in $R^{-1}(z)$, which also coincide with the case of non-correlated white noise MIMO channels reported in [14]. Theorem 3.1 shows that the conditions for stabilizability in this case depend on the interplay between the channel noise spectrum and the pole locations and directions.

Theorem 3.1 can be simplified under more special circumstances.

Corollary 3.1: Consider the setup stated in Theorem 3.1.

i) If the plant has only one unstable pole p with direction ξ, then the system is MSS and satisfies the SNR constraints if and only if

$$\max_{i\in\{1,2,\dots,n_y\}}\frac{(|p|^2-1)|\eta_i^\top\xi|^2}{\sigma_{q_i}^2\Gamma_i\|\Omega_q(p)^{-1}\xi\|^2}<1. \tag{10}$$

ii) Define $R_{n_y+1}^{-1}=I$. If the plant has n_y unstable poles $\{p_i\}_{i=1}^{n_y}$ with canonical directions $\{\eta_i\}_{i=1}^{n_y}$ respectively, then the system is MSS and satisfies the SNR constraints if and only if

$$\max_{i\in\{1,2,\dots,n_y\}}\frac{(|p_i|^2-1)\|V_i(p_i)\Omega_q^{-1}(p_i)\eta_i\|^2}{\|R_{i-1}(p_i)\cdots R_0(p_i)\Omega_q^{-1}(p_i)\eta_i\|^4\sigma_{q_i}^2\Gamma_i}<1, \tag{11}$$

where $V_i(z)=R_{n_y+1}^{-1}(z)\cdots R_{i+1}^{-1}(z)R_{i-1}(z)\cdots R_0(z)$.

Proof:

i) We compute $\mathcal{A}_1$ as

$$\mathcal{A}_1=\Omega_q(p)(1-|p|^2)\frac{\Omega_q(p)^{-1}\xi\xi^\top\Omega_q(p)^{-\mathsf{H}}}{\|\Omega_q(p)^{-1}\xi\|^2}.$$

The result is obtained directly by using Theorem 3.1 with $n_p=1$.

ii) We can write $\mathcal{A}_\ell$ as

$$\mathcal{A}_\ell=\frac{(1-|p_\ell|^2)\eta_\ell\eta_\ell^\top\Omega_q^{-\mathsf{H}}(p_\ell)V_\ell(p_\ell)^{\mathsf{H}}}{\|R_{l-1}(p_\ell)\cdots R_0(p_\ell)\Omega_q^{-1}(p_\ell)\eta_\ell\|^2}.$$

Since $\eta_i^\top\mathcal{A}_\ell=0$ for a fixed $i\neq\ell$, then the only non zero term of the double summation of (5) is $\frac{\eta_i^\top\mathcal{A}_i\mathcal{A}_i^{\mathsf{H}}\eta_i}{(|p_i|^2-1)\sigma_{q_i}^2\Gamma_i}$. After some tedious computations, the condition (5) leads to (11). ■

Equation (10) states that, as expected, stabilization under SNR constraints depends directly on how far from the unit circle a pole is. Also, the pole's direction plays a crucial role, as it makes clear that if the control effort of the unstable pole is concentrated on one particular channel, this channel must have greater SNR in order to stabilize the system. At last, the color and correlation influence is made explicit. The frequency response of the colored noise can increase the minimum SNR for stabilization, through the gain given by the inverse of $\Omega_q(z)$ at the plant unstable poles, which is further compounded by the pole direction. Also, if the noise q has high correlation among its components, then $\Omega_q(p)$ is close to singular and the SNR requirements drop drastically in at least one individual channel pair. In other words, since highly correlated noise can be analyzed as noise originated from a common source, stabilization effort will be reduced in at least one of the common noise channels, as more knowledge of the real output signal can be deduced.

On the other hand, Equation (11) shows that under noise correlation, even if all plant states can be measured independently, the poles of other channel components influence the SNR limits of each individual SISO channel. This is a property not seen in additive white noise channel configurations like the one in [13].

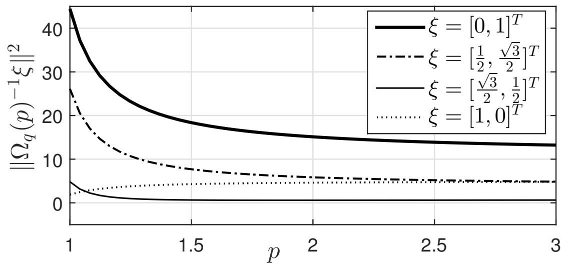

Fig. 2: Value of $\|\Omega_q(p)^{-1}\xi\|^2$ for different poles p and directions ξ.

In order to get further insight on the effect of noise color in the MSS conditions, an illustrative analysis of the behavior of $\|\Omega_q(p)^{-1}\xi\|^2$, is given next. We consider

$$\Omega_q(z) = \begin{bmatrix} \frac{z-0.7}{z-0.9} & 0 \\ 0.6 & \frac{0.3(z-0.8)}{z-0.6} \end{bmatrix},$$

and plot the magnitude $\|\Omega_q(p)^{-1}\xi\|^2$ versus p for different values of ξ (see Fig. 2). For the noise model proposed, the SNR requirements of every channel of a simple plant like Corollary 3.1.i) can change drastically depending on the pole's direction. Also, some convenient pole directions can mitigate the effect of high magnitude poles on the SNR limits for stabilization, as seen in with the direction $\xi = [1 \;\; 0]^\top$.

Non-colored noise stabilization: For more intuition into the effect of channel correlations, we assume next that the MIMO channel is correlated but white, and thus the noise covariance matrix P_q is given by $P_q = \Omega_q\Omega_q^T$, where the spectral factor Ω_q is now a constant invertible matrix. This allows us to state a simplified condition for mean square stabilizability.

Corollary 3.2: Assume that Assumption 2.1 holds, and $G(z)$ has relative degree 1 and no finite NMP zeros. Furthermore, assume that the noise q is white but correlated, that is, $\Omega_q(z) = \Omega_q$, with Ω_q a constant invertible matrix. Then, the feedback loop in Fig. 1 is MSS and the SNR constraints in (1) are satisfied if and only if

$$\max_{i\in\{1,\ldots,n_y\}} \frac{1}{\sigma_{q_i}^2\Gamma_i}\left(\eta_i^\top\Omega_q R^{-1}(\infty)R^{-1}(\infty)^{\mathsf{H}}\Omega_q^\top\eta_i - \sigma_{q_i}^2\right) < 1. \tag{12}$$

Proof: Note that in this case we simply have

$$\sum_{\ell=1}^{n_p}\frac{\mathcal{A}_\ell}{z-p_\ell} = \Omega_q(R^{-1}(z) - R^{-1}(\infty)).$$

From (9) we then obtain

$$\begin{aligned} J_i(Q_i^{\text{opt}}) &= \frac{1}{\sigma_{q_i}^2\Gamma_i}\Bigg(\|\eta_i^\top\Omega_q R^{-1}(\infty)\|_2^2 + \|\eta_i^\top\Omega_q R^{-1}(z)\|_2^2 \\ &\quad - \frac{2}{2\pi j}\oint_{z\in\mathbb{C}:|z|=1}\eta_i^\top\Omega_q R^{-1}(z)R^{-1}(\infty)^{\mathsf{H}}\Omega_q^\top\eta_i\frac{dz}{z}\Bigg) \\ &= \frac{1}{\sigma_{q_i}^2\Gamma_i}\left(\eta_i^\top\Omega_q R^{-1}(\infty)R^{-1}(\infty)^{\mathsf{H}}\Omega_q^\top\eta_i - \sigma_{q_i}^2\right). \end{aligned}$$

As a result, (12) follows from (3). ■

Corollary 3.2 shows that for correlated noise, the effect of unstable poles with their directions for MSS under SNR constraints can be studied in the $R^{-1}(\infty)$ matrix, similar to what has been obtained in [13], [14]. Intuitively, pole directions proportionally distribute the SNR requirements on each channel. However, in the current setup Ω_q can alter the directions used to construct $R(z)$ (see (4)), affecting the power allocation among channels. If Ω_q is diagonal, then the noise is white and no-correlated and the condition in (12) reduces to Lemma 2 of [14] for a given scaling matrix.

It is worth noting that the condition in Corollary 3.2 can be alternatively verified without obtaining the factor $R(z)$, but instead via a computational result. In this regard, we rewrite the MSS conditions in Corollary 3.2 based on the solution of a discrete time algebraic Riccati equation (DARE), a modified algebraic Riccati equation (MARE), and a linear matrix inequality (LMI). Before stating the result, we present a technical proposition.

Proposition 3.1: Assume that the pair (A, C) is observable and of suitable dimensions, and $P_q > 0$. Then, the following conditions are equivalent:

i) There exists a positive definite matrix $\mathcal{X} = \mathcal{X}^\top$ such that

$$\begin{cases} \mathcal{X} = A\mathcal{X}A^\top - A\mathcal{X}C^\top(P_q + C\mathcal{X}C^\top)^{-1}C\mathcal{X}A^\top \\ \gamma_i\sigma_i^2 > \eta_i^\top C\mathcal{X}C^\top\eta_i, \quad i = 1,2,\ldots n_y. \end{cases} \tag{13}$$

ii) There exists a positive definite matrix $\mathcal{X} = \mathcal{X}^\top$ such that

$$\begin{cases} \mathcal{X} \geq A\mathcal{X}A^\top - A\mathcal{X}C^\top(P_q + C\mathcal{X}C^\top)^{-1}C\mathcal{X}A^\top \\ \gamma_i\sigma_i^2 > \eta_i^\top C\mathcal{X}C^\top\eta_i, \quad i = 1,2,\ldots n_y. \end{cases} \tag{14}$$

iii) There exists a positive definite matrix $\mathcal{X} = \mathcal{X}^\top$ and a matrix J such that

$$\begin{cases} \mathcal{X} \geq (A + JC)\mathcal{X}(A + JC)^\top + JP_qJ^\top \\ \gamma_i\sigma_i^2 > \eta_i^\top C\mathcal{X}C^\top\eta_i, \quad i = 1,2,\ldots n_y. \end{cases} \tag{15}$$

Proof: See Appendix. ■

Theorem 3.2: Assume that G has a state space representation given by $(\text{diag}(A_s, A_u), [B_s^\top \;\; B_u^\top]^\top, [C_s \;\; C_u], 0)$, where A_s contains all stable poles of G, A_u contains all unstable poles, and where B_s, B_u, C_s and C_u are input and output matrices with appropriate dimensions. Define $\Gamma \triangleq \text{diag}(\Gamma_1, \Gamma_2, \ldots, \Gamma_{n_y})$. Then, there exists a controller $K(z)$ achieving MSS and satisfying the SNR constraints if and only if any of the following equivalent conditions are satisfied:

i) There exists a positive definite $\mathcal{X} = \mathcal{X}^\top$ such that, for every $i \in \{1,\ldots,n_y\}$

$$\eta_i^\top(\Gamma P_q - C_u\mathcal{X}C_u^\top)\eta_i > 0, \tag{16}$$

where $P_q = \Omega_q\Omega_q^\top$ and $\mathcal{X}$ is the unique stabilizing solution of the DARE

$$\mathcal{X} = A_u\mathcal{X}A_u^\top - A_u\mathcal{X}C_u^\top(P_q + C_u\mathcal{X}C_u^\top)^{-1}C_u\mathcal{X}A_u^\top. \tag{17}$$

ii) There exists a diagonal matrix $W > -(\Gamma^{-1}+I)\odot(P_q)$ such that the following MARE has a unique stabilizing solution $\mathcal{X} = \mathcal{X}^\top$:

$$\begin{aligned} \mathcal{X} =& A_u\mathcal{X}A_u^\top - A_u\mathcal{X}C_u^\top \\ &((\mathbf{1}+\Gamma^{-1})\odot(P_q + C_u\mathcal{X}C_u^\top) + W)^{-1}C_u\mathcal{X}A_u^\top, \end{aligned}$$

where $\mathbf{1}$ denotes the square matrix with all elements equal to 1.

iii) There exist matrices $S = S^\top > 0$ and V such that the following LMIs hold:

$$\begin{bmatrix} S & SA_u + VC_u & V \\ A_u^\top S + C_u^\top V^\top & S & 0 \\ V^\top & 0 & P_q^{-1} \end{bmatrix} > 0 \tag{18}$$

$$\begin{bmatrix} \gamma_i\sigma_i^2 & \eta_i^\top C_u \\ C_u^\top\eta_i & S \end{bmatrix} > 0, \qquad i = \{1,\ldots,n_y\}. \tag{19}$$

Proof:

i) From the factorization in (4), we can assume without loss of generality that $R(z)$ can be calculated using a coprime factorization with co-inner denominator for the plant $G_R(z) = \Omega_q^{-1}G(z)$. Thus, we can write $G_R(z) = R^{-1}(z)\tilde{N}_R(z)$ with $R(z)$ co-inner and $\tilde{N}_R(z) \in \mathcal{RH}_\infty$. It is well known that such co-inner factor can be calculated solving a DARE (see e.g. [22]). Based on such results, we have that

$$R(\infty) = (I + \Omega_q^{-1} C_u \mathcal{X} C_u^\top \Omega_q^{-\top})^{-1/2}, \tag{20}$$

where $\mathcal{X} = \mathcal{X}^\top > 0$ is the solution to the DARE

$$\begin{aligned} \mathcal{X} &= A_u \mathcal{X} A_u^\top \\ &\quad - A_u \mathcal{X} C_u^\top \Omega_q^{-\top} (I + \Omega_q^{-1} C_u \mathcal{X} C_u^\top \Omega_q^{-\top})^{-1} \Omega_q^{-1} C_u \mathcal{X} A_u^\top \\ &= A_u \mathcal{X} A_u^\top - A_u \mathcal{X} C_u^\top (P_q + C_u \mathcal{X} C_u^\top)^{-1} C_u \mathcal{X} A_u^\top. \end{aligned} \tag{21}$$

Considering (20) and Theorem 3.1, the stabilization with SNR constraints problem is equivalent to the condition that, for all $i \in \{1, \ldots, n_y\}$,

$$\frac{1}{\sigma_{q_i}^2 \Gamma_i} \left(\eta_i^\top \Omega_q (I + \Omega_q^{-1} C_u \mathcal{X} C_u^\top \Omega_q^{-\top}) \Omega_q^\top \eta_i - \sigma_{q_i}^2 \right) < 1,$$

where $\mathcal{X}$ is the solution of the DARE in (17). These conditions lead to (16).

ii) Given (16), there exists a diagonal matrix $\tilde{W} > 0$ such that

$$\tilde{W} - I \odot P_q + \Gamma^{-1} \odot (C_u \mathcal{X} C_u^\top) = 0.$$

Including this expression in the parenthesis of (17),

$$\begin{aligned} \mathcal{X} = A_u \mathcal{X} A_u^\top - A_u \mathcal{X} C_u^\top \Omega_q^{-\top} (\tilde{W} - (\Gamma^{-1} + I) \odot P_q \\ + (\mathbf{1} + \Gamma^{-1}) \odot (P_q + C_u \mathcal{X} C_u^\top))^{-1} C_u \mathcal{X} A_u^\top \end{aligned}$$

If we define $W = \tilde{W} - (\Gamma^{-1} + I) \odot P_q$, we have the stated result.

iii) Now, suppose that (16) and (17) hold for some $\mathcal{X}$ and J. Applying Schur Complement condition for positive definiteness [23], we write the inequalities as follows:

$$\begin{bmatrix} \mathcal{X} & A_u + J C_u & J \\ (A_u + J C_u)^\top & \mathcal{X}^{-1} & 0 \\ J^\top & 0 & P_q^{-1} \end{bmatrix} > 0$$

$$\begin{bmatrix} \gamma_i \sigma_i^2 & \eta_i^\top C_u \\ C_u^\top \eta_i & \mathcal{X}^{-1} \end{bmatrix} > 0, \qquad i = \{1, \ldots, n_y\}.$$

Letting $S = \mathcal{X}^{-1}$ and $V = \mathcal{X}^{-1} J$, and pre- and post-multiplying the matrix $\mathrm{diag}(\mathcal{X}^{-1}, I, I)$ on both sides on the first inequality above, respectively, leads to the inequalities (18) and (19). Therefore, a feasibility search considering these inequalities is equivalent to studying the existence of a controller such that MSS is achieved under certain fixed SNR constraints. ■

Theorem 3.2 permits us to determine numerically if a given MIMO minimum phase plant can be stabilized under fixed SNR constraints on white correlated noises channels. These conditions are given in three different forms: a constrained algebraic Riccati equation, a modified algebraic Riccati equation, and a LMI feasibility problem.

Remark 3.1: The optimization problem solved above for the correlated noise case is equivalent to minimizing the plant output norm of a NCS with fixed channel scaling matrix Ω_q^{-1}, see Fig. 3 b). In general terms, minimizing y does not imply optimal channel SNR for this configuration, as the minimization should be on v, not on y.

IV. Stabilizability of Non Minimum-phase plants

Now we return to the case with colored and correlated noise channels and extend the condition in Theorem 3.1 to a larger class of MIMO systems. For this matter, we add the following assumptions on $G(z)$:

Assumption 4.1:

- The NMP zeros of $G(z)$ have canonical output directions.

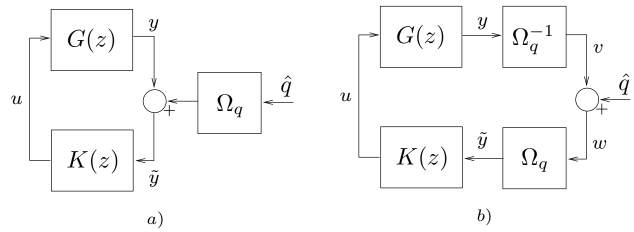

Fig. 3: Alternative interpretation a) Correlated noise architecture. b) White noise architecture, with channel scaling matrix Ω_q^{-1}.

- The unstable poles and finite NMP zeros of $G(z)$ have multiplicity one.

For the following result, we use the decomposition given in (7), and write

$$\nabla(z) = \sum_{\tau=0}^{\infty} \Phi_\tau z^{-\tau}. \tag{22}$$

Theorem 4.1: Consider the NCSs depicted in Fig. 1, and assume that Assumptions 2.1 and 4.1 hold. We denote $\{p_i\}_{i=1}^{n_p}$ and $\mathcal{C}_i \triangleq \{c_{i_1}, \ldots, c_{i n_{c_i}}\}$ as the set of distinct unstable poles of $G(z)$ and the set of finite NMP zeros of $\eta_i^\top G(z)$ respectively. Also, r_i is the number of zeros at the infinity of $\eta_i^\top G(z)$. Then, the feedback loops in Fig. 1 are MSS and the SNR constraints in (1) are satisfied if and only if

$$\max_{i \in \{1,2,\ldots,n_y\}} \frac{\beta_i^{(1)} + \beta_i^{(2)} + \beta_i^{(3)}}{\sigma_{q_i}^2 \Gamma_i} < 1,$$

where

$$\beta_i^{(1)} \triangleq \sum_{\tau=1}^{r_i - 1} \|\eta_i^\top \Phi_\tau\| + \eta_i^\top \sum_{\ell=1}^{n_p} \sum_{k=1}^{n_p} \frac{\mathcal{A}_\ell \mathcal{A}_k^{\mathsf{H}}}{p_\ell p_k^* - 1} \eta_i, \tag{23}$$

$$\beta_i^{(2)} \triangleq \sum_{k=1}^{n_{c_i}} \sum_{j=1}^{n_{c_i}} \frac{\Theta_{i_k} \Theta_{i_j}^{\mathsf{H}}}{c_{i_k} c_{i_j}^* - 1}, \tag{24}$$

$$\beta_i^{(3)} \triangleq \sum_{k=1}^{r_i - 1} \left\| \sum_{j=1}^{n_{c_i}} \Theta_{i_j} c_{i_j}^{k-1} \right\|_2^2, \tag{25}$$

where $\{\mathcal{A}_l\}_{l=1}^{n_p}$ and $\{\Phi_\tau\}_{\tau=1}^{\infty}$ are obtained by (6) and (22) respectively, and

$$\Theta_{i_\ell} \triangleq \eta_i^\top \left(\sum_{\tau=r_i}^{\infty} \Phi_\tau c_{i_\ell}^{-\tau} \right) (1 - |c_{i_\ell}|^2) \prod_{\substack{j=1 \\ j \neq l}}^{n_{c_i}} \frac{1 - c_{i_j}^* c_{i_\ell}}{c_{i_\ell} - c_{i_j}}. \tag{26}$$

Proof: Similarly as in the proof of Theorem 3.1, we can express $J_i(Q_i)$, the functional to minimize over $Q_i \in \mathcal{RH}_\infty$, as

$$J_i(Q_i) = \frac{1}{\sigma_{q_i}^2 \Gamma_i} \|\eta_i^\top (X \tilde{M}_m - \Omega_q R^{-1} - N Q_i \tilde{M}_m)\|_2^2.$$

We define $\Psi_i \triangleq \eta_i^\top X \tilde{M}_m - \eta_i^\top \Delta - \sum_{\tau=0}^{r_i - 1} \eta_i^\top \Phi_\tau z^{-\tau}$. Note that $\Psi_i \in \mathcal{RH}_2$, and can be written as

$$\Psi_i = \eta_i^\top N \tilde{Y} \Omega_q R^{-1} + \sum_{\tau=r_i}^{\infty} \eta_i^\top \Phi_\tau z^{-\tau},$$

thus, Ψ_i has at least r_i zeros at the infinity. After some algebraic manipulation we obtain

$$J_i(Q_i) = \frac{1}{\sigma_{q_i}^2\Gamma_i}\left\|\Psi_i - \eta_i^\top\left(\nabla - \sum_{\tau=0}^{r_i-1}\Phi_\tau z^{-\tau} + NQ_i\tilde{M}_m\right)\right\|_2^2$$
$$= \frac{1}{\sigma_{q_i}^2\Gamma_i}\left(\|\Psi_i - \eta_i^\top NQ_i\tilde{M}_m)\|_2^2 + \beta_i^{(1)}\right),$$

where $\beta_1^{(1)}$ is defined as in (23).

It is well known that the set of NMP zeros of $\eta_i^\top N$ are the NMP zeros of $\eta_i^\top G = \eta_i^\top NM^{-1}$. With this fact in mind, we define the unitary function

$$\alpha_i \triangleq \prod_{\ell=1}^{n_{c_i}}\left(\frac{1-c_{i_\ell}^* z}{z-c_{i_\ell}}\right), \tag{27}$$

and decompose $\Psi_i\alpha_i$ as $\Psi_i\alpha_i = \lambda_i + \theta_i$ where $\lambda_i \in \mathcal{RH}_2$ and $\theta_i \in \mathcal{RH}_2^\perp$. Note that the set of unstable poles of $\Psi_i\alpha_i$ is given by $\mathcal{C}_i$. So, it is possible to write θ_i as

$$\theta_i = \sum_{\ell=1}^{n_{c_i}}\frac{\Theta_{i_\ell}}{z-c_{i_\ell}} = \sum_{\ell=1}^{n_{c_i}}\Theta_{i_\ell}\sum_{k=1}^{\infty}c_{i_\ell}^{k-1}z^{-k}, \tag{28}$$

where Θ_{i_ℓ} is as in (26). Therefore, we have

$$J_i(Q_i) = \frac{1}{\sigma_{q_i}^2\Gamma_i}\left(\left\|\lambda_i - \eta_i^\top\alpha_i NQ_i\tilde{M}_m\right\|_2^2 + \|\theta_i\|_2^2 + \beta_i^{(1)}\right)$$
$$= \frac{1}{\sigma_{q_i}^2\Gamma_i}\left(\left\|\lambda_i - \eta_i^\top\alpha_i NQ_i\tilde{M}_m\right\|_2^2 + \beta_i^{(1)} + \beta_i^{(2)}\right),$$

where $\beta_i^{(2)}$ has been calculated using standard norm-2 techniques and is given by (24). By considering the fact that Ψ_i has at least r_i zeros at the infinity, we can write $J_i(Q_i)$ as

$$J_i(Q_i) = \frac{1}{\sigma_{q_i}^2\Gamma_i}\Bigg(\beta_i^{(1)} + \beta_i^{(2)} + \beta_i^{(3)}$$
$$+ \left\|\Psi_i\alpha_i - \sum_{\ell=1}^{n_{c_i}}\Theta_{i_\ell}\sum_{k=r_i}^{\infty}c_{i_\ell}^{k-1}z^{-k} - \eta_i^\top\alpha_i NQ_i\tilde{M}_m\right\|_2^2\Bigg) \tag{29}$$

where

$$\beta_i^{(3)} = \left\|\sum_{\ell=1}^{n_{c_i}}\Theta_{i_\ell}\sum_{k=1}^{r_i-1}c_{i_\ell}^{k-1}z^{-k}\right\|_2^2, \tag{30}$$

which is equivalent to the expression (25). Given (29), it is possible to choose $Q_i = Q_i^{\text{opt}} \in \mathcal{RH}_\infty$, such that $J_i^{\text{opt}} = \frac{\beta_i^{(1)}+\beta_i^{(2)}+\beta_i^{(3)}}{\sigma_{q_i}^2\Gamma_i}$. Our claim follows from (3). ■

Theorem 4.1 states necessary and sufficient conditions for stabilizability under SNR constraints that hold for a wide range of unstable and non-minimum phase MIMO systems. As expected, the noise color plays an essential role in the stabilization conditions. The system's SNR requirements for mean square stability depend on the relation between the location of the NMP zeros, unstable poles and the noise's spectrum, as seen in the terms Θ_{i_ℓ}. The previously studied pole-noise spectrum relationship naturally plays a role as well. The requirements increase by larger relative degrees r_i, due to $\beta_i^{(3)}$ and the first term of $\beta_i^{(1)}$.

As a corollary, we show a explicit condition for a simple case in which the channel noise is white but correlated between channels.

Corollary 4.1: Consider the setup in Theorem 4.1 but with Ω_q a non-singular constant matrix. Also, consider the plant $G(z)$ of relative degree 1, with one pole $p \in \mathbb{R} : |p| > 1$ with direction ξ, and n_y non-minimum phase zeros $\{c_i\}_{i=1}^{n_y}$ with canonical directions, one for each $\eta_i G(z), i = 1, \dots, n_y$. Then, the feedback loops in Fig. 1 are MSS and the SNR constraints in (1) are satisfied if and only if

$$\frac{(p^2-1)(c_i p-1)^2|\eta_i^\top\xi|^2}{\sigma_{q_i}^2\Gamma_i(c_i-p)^2\|\Omega_q^{-1}\xi\|^2} < 1 \quad, i = 1, 2, \dots, n_y. \tag{31}$$

Proof: From Theorem 4.1 we calculate $\beta_i^{(1)} = \frac{(p^2-1)|\eta_i^\top\xi|^2}{\|\Omega_q^{-1}\xi\|^2}$, $\beta_i^{(2)} = \frac{(c_i^2-1)(p^2-1)^2|\eta_i^\top\xi|^2}{(c_i-p)^2\|\Omega_q^{-1}\xi\|^2}$, and $\beta_i^{(3)} = 0$. ■

From (31) we note that the magnitude of the NMP zeros limit the stabilizability region given a fixed SNR constraint. Also, stability can be achieved only under high SNR limits if the unstable pole and NMP zero are closely located to each other. Finally, like the condition (10) written above, correlation between channels may reduce the SNR limits for stability, depending on how much correlation there is between channel noises.

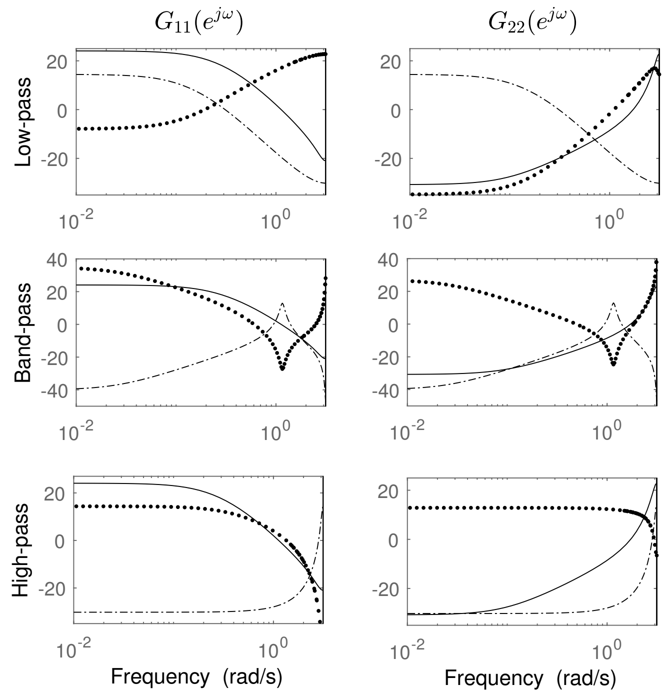

Fig. 4: Bode plots of the plant diagonal entries G_{11} and G_{22} (solid, first and second columns respectively), noise spectra (dash-dot, low, band and high-pass in each row respectively), and optimal transfer function T_{qy} (dotted).

V. Numerical Examples

In this section we illustrate our results with 2 numerical examples.

Example 5.1: We first study the effect of noise color in the SNR constraints for stabilization. Consider a 2x2 plant[1] given by $G = \text{diag}\left(\frac{z+0.6}{z(z-0.8)(z-1.5)}, \frac{2(z-1.1)}{(z+1.3)(z+2)}\right)$ and the following three cases for noise spectra i) a low pass $\Omega_q^{lp} = 0.106 I_{2\times2}/(z^2 - 1.7z + 0.72)$, ii) a band-pass $\Omega_q^{bp} = 0.229(z^2 - 0.95)I_{2\times2}/(z^2 - 0.76z + 0.9)$ and iii) a high pass $\Omega_q^{hp} = 0.106 I_{2\times2}/(z^2 + 1.7z + 0.72)$. Note that $G_{11}(z)$ and $G_{22}(z)$ are low and high pass transfer functions respectively.

[1]For this example we have considered a decentralized transfer function in order to analyze the effects of noise's frequency responses with more clarity. Similar results can be obtained with more general plants.

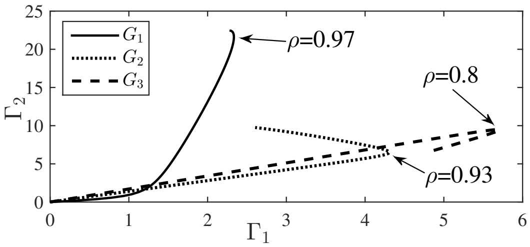

Fig. 5: Minimum SNR limits for stabilization of minimum phase plants $G_i(z)$, while varying the correlation coefficient ρ from 0 to 1.

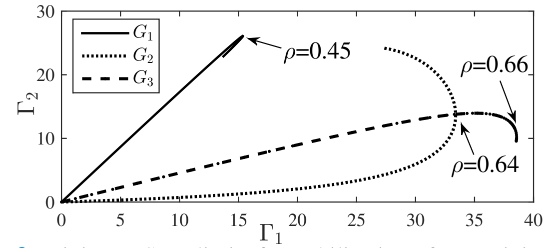

Fig. 6: Minimum SNR limits for stabilization of non-minimum phase plants $G_i(z)$, while varying the correlation coefficient ρ from 0 to 1.

In Fig. 4 we plot the frequency response of the plant's diagonal entries, the noise spectra (low-pass, band-pass, and high-pass filters), and the optimal transfer function T_{qy} obtained from (2) for each case. Note that in general, the optimal controller tends to attenuate the frequencies where the noise spectra are present with the most energy. This is a natural conclusion, as the minimization forces T_{qy} to optimally block the colored noise q.

The SNR limits for each noise spectra are given in Table I. From Table I, we conclude that less SNR is required on the channels where the noise is attenuated by the channel's SISO plant.

TABLE I: Minimum SNR necessary for MSS.

$\Gamma_i \backslash \Omega_q$	Ω_q^{lp}	Ω_q^{bp}	Ω_q^{hp}
Γ_1	0.8988	0.0619	0.0052
Γ_2	0.0223	0.1335	0.6174

Example 5.2: Consider three plants given by the state space representations $G_i = (A, B, C_i, 0_{2\times 2})$, where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} -0.42 & -0.09 \\ -1.23 & 0 \\ 0 & 1.42 \end{bmatrix},$$
$$C_i = \begin{bmatrix} -1.8 & -1.84 & 0.58 \\ 0 & k_i & 1.4 \end{bmatrix}, k_i \in \{-3, -1, 3\}. \quad (32)$$

These plants have two unstable poles at $p_1 = 3$ and $p_2 = 2$, with directions $\xi_1 = \begin{bmatrix} 0.38 & 0.92 \end{bmatrix}^\top$ and $\xi_2 = \begin{bmatrix} -1.84 & k_i \end{bmatrix}^\top / \sqrt{3.39 + k_i^2}$ respectively, where $k_i \in \{-3, -1, 3\}$ is left as a parameter to analyze the effect of this pole's direction. All plants have a minimum phase zero in $z = 0.5$, and relative degree 1. Also, consider a white correlated noise q with covariance matrix given by

$$P_q = \begin{bmatrix} 2 & \sqrt{6}\rho \\ \sqrt{6}\rho & 3 \end{bmatrix}, \rho \in [0, 1]. \quad (33)$$

where ρ is the correlation coefficient between the noise components. For every fixed k_i, we vary ρ in order to plot the effect of correlation in the the minimum SNR requirements, for different pole directions.

The results are shown in Fig. 5. Each segment is plotted from no correlation (top end of each segment) to full correlation (i.e. $\rho = 1$), where Ω_q is singular. For this degenerated case, all the systems can be stabilized without requirements of SNR on all channels. This result can be understood intuitively by the denominator of the conditions (10). It is seen that for a general pole direction, small noise correlation will worsen the SNR requirements and high to very high noise correlation will lower the SNR requirements. In this case, the curves in Fig. 5 show that Γ_1 and Γ_2 can increase as ρ decreases but, from some value onwards (printed in the graphics), all the SNR limits decrease.

We now include a non-minimum phase zero in our analysis. The plants have the same poles as before, but with a non-minimum phase zero in $z = 1.5$ with canonical direction. The state-space matrices are the following:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} -0.89 & -0.14 \\ -2.45 & 0 \\ 0 & 1.45 \end{bmatrix},$$
$$C_i = \begin{bmatrix} -3.35 & -0.41 & 0.34 \\ 0 & k_i & 1.38 \end{bmatrix}, k_i \in \{-3, -1, 3\}. \quad (34)$$

The directions of the unstable poles $z = 3$ and $z = 2$ are $\xi_1 = \begin{bmatrix} 0.24 & 0.97 \end{bmatrix}^\top$ and $\xi_2 = \begin{bmatrix} -0.41 & k_i \end{bmatrix}^\top / \sqrt{0.17 + k_i^2}$ respectively. Note that in this case, the pole direction ξ_2 changes more abruptly between plants. We again plot the minimum SNR requirements for each plant, with the same covariance matrix (33) and ρ variation. The results are shown in Fig. 6. An interesting observation is that the requirements on the second individual channel (i.e. Γ_2), have essentially not changed in magnitude compared to the minimum plant examples. An increment of SNR requirement is perceived only in Γ_1 for all cases, since the NMP zero is present on $\eta_1^\top G(z)$. Again as expected, high correlation will lower the SNR requirements for all channels for mean square stabilization.

VI. Conclusions

This paper addressed the problem of mean square stabilization of MIMO discrete-time LTI plants over additive colored and mutually correlated channels. Explicit conditions for MSS under individual independent SNR constraints have been obtained for unstable minimum and non-minimum phase plants, in which the effects of noise color and correlation in stabilizability have been revealed. We also obtained simplified conditions for several particular cases, in which we found the explicit relationship between the noise spectrum, unstable poles, NMP zeros, relative degree and SNR limits, that must hold for stabilizability. Examples exposing colored and mutually correlated noise cases have been put forward showing the theoretical findings.

Appendix: Proof of Proposition 3.1

Proof:

$i) \implies ii)$: Straightforward.

$ii) \implies i)$: (14) implies that there exists a semidefinite matrix $Q \geq 0$ such that

$$\mathcal{X} = A\mathcal{X}A^\top - A\mathcal{X}C^\top(P_q + C\mathcal{X}C^\top)^{-1}C\mathcal{X}A^\top + Q \quad (35)$$

If we denote $\tilde{\mathcal{X}}$ as the solution of the DARE in (13), by the comparison Theorem 13.3.1 of [24] we have $\tilde{\mathcal{X}} \leq \mathcal{X}$, hence the inequalities of (13) are also satisfied for $\tilde{\mathcal{X}}$.

$ii) \implies iii)$: Let $J = -A\mathcal{X}C^\top(P_q + C\mathcal{X}C^\top)^{-1}$. Then, the inequality in (14) can be written as

$$\begin{aligned} \mathcal{X} &\geq A\mathcal{X}A^\top - A\mathcal{X}C^\top(P_q + C\mathcal{X}C^\top)^{-1}C\mathcal{X}A^\top \\ &= A\mathcal{X}A^\top + A\mathcal{X}C^\top J^\top + JC\mathcal{X}C^\top J^\top + JC\mathcal{X}A^\top + JP_qJ^\top \\ &= (A + JC)\mathcal{X}(A + JC)^\top + JP_qJ^\top. \end{aligned}$$

$iii) \implies ii)$: Again, we consider the matrix $J_1 = -A\mathcal{X}C^\top(P_q + C\mathcal{X}C^\top)^{-1}$. Note that

$$\begin{aligned}&(A+JC)\mathcal{X}(A+JC)^\top + JP_qJ^\top \\ &\quad -(A+J_1C)\mathcal{X}(A+J_1C)^\top - J_1P_qJ_1^\top \\ &= (J-J_1)(P_q + C\mathcal{X}C^\top)(J-J_1)^\top \geq 0. \end{aligned} \tag{36}$$

Thus, repeating the steps of the $ii) \implies iii)$ part of this proof,

$$\begin{aligned}\mathcal{X} &\geq (A+JC)X(A+JC)^\top + JP_qJ^\top \\ &\geq (A+J_1C)\mathcal{X}(A+J_1C)^\top + J_1P_qJ_1^\top \\ &= A\mathcal{X}A^\top - A\mathcal{X}C^\top(P_q + C\mathcal{X}C^\top)^{-1}C\mathcal{X}A^\top, \end{aligned} \tag{37}$$

obtaining the condition in (14). ■

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