# On the existence of a stabilizing solution of Modified Algebraic Riccati Equations in terms of standard Algebraic Riccati Equations and Linear Matrix Inequalities

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*Abstract*—**In this paper, we study conditions for the existence of stabilizing solutions of two classes of modified discrete algebraic Riccati equations (MAREs) emerging in stochastic control problems. In order to do so, we first rewrite each MARE in terms of a standard ARE subject to specific constraints, which allows us to connect their solution with a set of linear-control constrained problems. With this result we also determine, for each MARE, a linear matrix inequality problem whose feasibility is guaranteed if and only if a stabilizing solution of the original MARE exists.**

*Index Terms*—**Modified algebraic Riccati equation, mean-square stabilizing solution, linear matrix inequalities, stochastic control**

## I. Introduction

In recent years, modified algebraic Riccati equations (MAREs) have gained attention as they have been encountered recurrently in Networked Control Systems and similar topics where feedback control problems subject to communication constraints are studied (see e.g. [1], [2], [3], [4]). Interestingly, first contributions on the analysis of MAREs appeared decades ago [5].

In this paper we are interested in studying the existence of solutions of discrete-time MAREs of the form

$$X = A^\top XA - A^\top XB[W \odot (B^\top XB + R)]^{-1}B^\top XA + Q, \tag{1}$$

where $A, B, X, W, R$ and $Q$ are matrices of appropriate dimensions. As the name suggests, MAREs are similar in structure to a standard algebraic Riccati equation (ARE); however, they present an additional matrix parameter $W$ that may endanger the existence of admissible solutions.

It is well known that standard AREs have great importance in control theory since they are strongly related with linear quadratic control problems and estimation theory. Due to this, AREs have been extensively studied in several works (see e.g. [6] and references therein), and necessary and sufficient conditions for the existence of solutions that lead to stability of the underlying feedback control problem have been found. Such solutions are called stabilizing solutions. In a similar fashion, stabilization problems in several different scenarios have been found to depend on the existence of particular solutions of some MARE.

Mean square stabilization of feedback systems with additive channels subject to signal to noise ratio constraints have been analyzed in, e.g., [7] and [4], where necessary and sufficient conditions for stability are obtained which can be written in terms of the existence of a MARE. In [8], feedback control systems are studied considering multiplicative channels instead of additive ones for the communication between plant and controller. In such scenario, the mean square stability of the loop also depends on the solution of a specific MARE. In [9], [10], [11], estimation problems under intermittent observations are considered under different setups. It has been shown there that the stability of such estimators also depends on the solution of a MARE. In the context of LQG under data dropouts, it has been shown that the existence of a stationary optimal estimator and the optimal controller in these setups depend on two different MAREs [1], [12], [2]. MAREs also appear for consensus problems in multi-agent systems (that can be viewed as a stability problem) under different frameworks such as observed-based protocols [13] or saturation constraints and synchronization requirements [3].

Here we are interested in studying the existence of a stabilizing solution of MAREs with the form in (1). The existence of a stabilizing solution has been studied also in previous works for different values of $W$. In [9], a MARE with $W$ only depending on one parameter $\gamma$ is studied. It was found that there exists a critical value of $\gamma$ in which a stabilizing solution exists, and an explicit expression for such critical value is given for some special cases. Also, a numerical method to determine the existence of stabilizing solutions is given in terms of linear matrix inequalities (LMIs). An alternative algorithm is given in [14] for the same type of MARE in [9]. In [15], an analytic expression is given for the MARE in [9] for a special case. However, such expression is rather complicated and hard to analyze, which is expected due to the difficulty of the problem. In [16], a more complicated MARE in which $W$ depends on several parameters is analyzed, which is inspired by the control problem studied in [8]. Conditions for the existence of a stabilizing solution are given in [16] based on cone-invariant operators. LMI-based conditions have also been given for a

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similar type of MAREs (see [12]).

In this work we study two kind of MAREs found in the literature and show that, regardless of the original problem that give rise of such type of MARE, we can always rewrite it as a standard ARE subject to specific constraints. This has two advantages: on one hand, it permits us to give an alternative look to the original problem as a standard control problem with constraints, and on the other hand, it allows us to exploit the well known properties of standard AREs to determine conditions for the existence of stabilizing solutions. As a second contribution, we present conditions for the existence of stabilizing solutions for the aforementioned type of MAREs in terms of a LMI feasibility problem that can be easily checked with computational tools. As mentioned before, some LMI conditions have been given in previous works in different contexts (see e.g. [9], [12]), however we derive them in this work considering a unified framework based on AREs, which is also valid for previous works with MAREs that fit the structure in (1).

The structure of the paper is as follows. In Section II, the two types of MAREs considered in this paper are presented. The first type of MARE is studied in Section III, where both alternative forms based on AREs and LMIs are given. Analogous results are found in Section IV for the second type of MARE. Some examples are given in Section V, and final conclusions can be found in Section VI.

*Notation:* The $n \times n$ identity matrix is denoted by $I_{n\times n}$, and the diagonal matrix $\Lambda$ with entries $\Lambda_{ii} = \lambda_i$ is written as $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. The symbol $\odot$ denotes the Hadamard product, $\cdot^\top$ is used to denote transpose and $1_{m\times m}$ is defined as the $m \times m$ matrix with only ones at its entries.

## II. Problem Setup

In this paper we study necessary and sufficient conditions for the existence of a stabilizing and positive definite solution $X = X^\top$ of the MARE in (1), where $A, X, Q \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $R, W \in \mathbb{R}^{m\times m}$.

A MARE of the form (1) has the structure of a standard discrete time Algebraic Riccati Equation (ARE) but with an additional matrix parameter $W$, which has element-wise influence on $B^\top XB + R$. A trivial case is when $W = 1_{m\times m}$, in which the MARE becomes the standard ARE:

$$X = A^\top XA - A^\top XB(R + B^\top XB)^{-1}B^\top XA + Q. \quad (2)$$

It is worth to recall that the ARE in (2) has been well studied due to the key role played in optimal control and estimation problems. It is also well known that, for $R \succ 0$, $Q \succeq 0$, the ARE in (2) has a positive definite and stabilizing solution if and only if the pair $(A, B)$ is controllable [6].

It is expected that the existence of a stabilizing solution of (1) depends on the matrix $W$. In this paper we study two different types of MARE found in the literature.

The first case is the MARE in (1) when

$$W = 1_{m\times m} + \Gamma^{-1},$$

for $Q \succeq 0$ and $\Gamma, R, \succ 0$ with $R$ and $\Gamma$ diagonal matrices. Thus, the $m$ positive parameters in $\Gamma$ are those that affect the existence of stabilizing solutions. Clearly, if the elements in $\Gamma$ grow unbounded, the MARE becomes an ARE. Such type of MARE can be found, for instance, in [12], [8], [2].

The second type of MARE analyzed in this paper corresponds to the MARE in (1) for

$$W = 1_{m\times m}/\gamma,$$

with $R \succ 0$, $Q \succeq 0$ and $\gamma \in (0, 1)$. Evidently, for $\gamma = 1$, the MARE becomes an ARE. Such kind of MAREs have appeared, for instance, in [9], [10], [3].

Our goal is to establish necessary and sufficient conditions to ensure the existence of a stabilizing solution of both types of MAREs. These MAREs have appeared in literature related with control problems involving stochastic processes, thus a *stabilizing solution* to the MARE indeed refers to a *mean square stabilizing solution* of the underlying control problem. Nevertheless, for simplicity we use *stabilizing solution* instead of *mean square stabilizing solution*.

It is also important to note that the analysis made in this paper is also valid for the dual MARE

$$X = AXA^\top - AXC^\top[W \odot (CXC^\top + R)]^{-1}CXA^\top + Q,$$

which can be found frequently in estimation problems (e.g. [9]).

## III. Case with $W = 1_{m\times m} + \Gamma^{-1}$

For this case, the MARE in (1) can be written as

$$X = A^\top XA - A^\top XB\Big[B^\top XB + \Gamma^{-1} \odot B^\top XB + (I + \Gamma^{-1})R\Big]^{-1} B^\top XA + Q, \quad (3)$$

with $Q \succeq 0$ and

$$R = \mathrm{diag}(r_1, r_2, \ldots, r_m) \succ 0,$$
$$\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_m) \succ 0.$$

### A. MARE as a standard ARE subject to constraints

In this section we show that the set of MAREs in (3) for $W = 1_{m\times m} + \Gamma^{-1}$ can always be written as a standard Riccati equation problem subject to a specific restriction. This will help us to establish, in the next subsection, conditions for the existence of stabilizing solution of such MAREs in terms of LMIs.

*Lemma 3.1:* Assume that the pair $(A, B)$ is controllable. Denote by $\eta_i$ the canonical vector such that $[\eta_1\, \eta_2 \cdots \eta_m] = I_{m\times m}$. The MARE in (3) has a stabilizing solution if and only if there exists a positive definite matrix $X$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \succ 0$ such that, for $i = 1, 2, \ldots m$

$$\gamma_i\lambda_i - (1 + \gamma_i)r_i - \eta_i^\top B^\top XB\eta_i \geq 0, \quad (4)$$

where $X$ is a stabilizing solution of the ARE

$$X = A^\top XA - A^\top XB(B^\top XB + \Lambda)^{-1}B^\top XA + Q. \quad (5)$$

*Proof:* First, if we introduce the auxiliary variable

$$\Lambda = \Gamma^{-1} \odot (B^\top XB + R) + R, \quad (6)$$

it is clear from (3) and (6) that solving the MARE is equivalent to solving the following ARE with constraints

$$\begin{cases} X = A^\top XA - A^\top XB(B^\top XB + \Lambda)^{-1}B^\top XA + Q, \\ \Lambda = \Gamma^{-1} \odot (B^\top XB) + \Gamma^{-1}R + R, \\ X \succ 0. \end{cases} \tag{7}$$

Thus, if (7) holds for some $X$ and $\Lambda \succ 0$, then (4) and (5) are readily satisfied with these matrices.

For the converse, we assume that the matrix pair $(X_0, \Lambda_0)$ satisfies equations (4) and (5). Thus,

$$\Lambda_0 \succeq \Gamma^{-1} \odot (B^\top X_0 B) + \Gamma^{-1}R + R. \tag{8}$$

We now define $\Lambda_1$ equal to the right hand side of (8). Like this, we have that $0 \prec \Lambda_1 \preceq \Lambda_0$. Since the pair $(A, B)$ is controllable, it follows from Theorem 13.3.1 in [6] that the ARE

$$X = A^\top XA - A^\top XB(B^\top XB + \Lambda_1)^{-1}B^\top XA + Q.$$

has stabilizing solution $X = X_1 \preceq X_0$. Subsequently, we can construct $\Lambda_2 \preceq \Lambda_1$ as the right hand side of (8) with $X_1$ instead of $X_0$. Repeating the process above, we obtain a sequence $\{\Lambda_0, \Lambda_1, \Lambda_2, \dots\}$ satisfying $\Lambda_0 \succeq \Lambda_1 \succeq \Lambda_2 \succeq \dots \succ \Gamma^{-1}R + R$. The sequence of positive definite solutions $\{X_0, X_1, X_2, \dots\}$ satisfies $X_0 \succeq X_1 \succeq X_2 \succeq \dots \succ 0$ and therefore both sequences converge to $\bar{\Lambda}$ and $\bar{X}$ respectively. These matrices satisfy

$$\begin{cases} \bar{X} = A^\top \bar{X}A - A^\top \bar{X}B(B^\top \bar{X}B + \bar{\Lambda})^{-1}B^\top \bar{X}A + Q, \\ \bar{\Lambda} = \Gamma^{-1} \odot (B^\top \bar{X}B) + \Gamma^{-1}R + R, \\ \bar{X} = \bar{X}^\top \succ 0. \end{cases}$$

The proof concludes by noting that, since $\Gamma^{-1}$ is diagonal, we can write the second restriction above as $\Gamma\bar{\Lambda} = I \odot (B^\top \bar{X}B) + \Gamma R$ and, given that $I \odot (B^\top \bar{X}B)$ is also diagonal, this constraint can be written as the set of $m$ constraints in (4). $\square$

Lemma 3.1 states that it is possible to express the MARE as an ARE whose solution is subject to a particular constraint. From a control theory perspective, it is well known that the term $B^\top XB$, where $X$ is the solution of (5), is closely related to the optimal cost of standard problems with quadratic performance criteria. For a stochastic control problem, $B^\top XB$ (or $CXC^\top$ for the dual case) could represent the stationary covariance matrix of a performance signal, and thus $\eta_i^\top B^\top XB\eta_i$ the stationary variance of the $i-$th element of such signal. Therefore, regardless of the nature of the underlying problem that may give rise to a MARE with the structure in (3), the result in Lemma (5) permits us to study the existence of a stabilizing solution based on an auxiliary and known stochastic control problem but with an additional constraint on $\eta_i^\top B^\top XB\eta_i$.

## *B. Conditions in terms of linear matrix inequalities*

In this section we exploit the previous result and the fact that we can relax the equality conditions concerning $\Lambda$ in (19) to an inequality. This is particularly useful for establishing an LMI condition, as shown next.

*Theorem 3.1:* Assume the pair $(A, B)$ is controllable. The MARE in (3) has a positive definite and stabilizing solution if and only if any of the following conditions are met:

i) There exist $X \succ 0$, $\Lambda \succ 0$ diagonal and $F \in \mathbb{R}^{m\times n}$ such that

$$X \succ (A + BF)^\top X(A + BF) + F^\top \Lambda F + Q, \tag{9}$$

$$\eta_i^\top B^\top XB\eta_i \leq \gamma_i\lambda_i - (1 + \gamma_i)r_i, \quad i = 1, \dots, m. \tag{10}$$

ii) There exist $S \succ 0$, $\Pi \succ 0$ diagonal and $Y$ such that

$$\begin{bmatrix} S & SA^\top + Y^\top B^\top & Y^\top & S\Omega^\top \\ AS + BY & S & 0 & 0 \\ Y & 0 & \Pi & 0 \\ \Omega S & 0 & 0 & I \end{bmatrix} \succ 0, \tag{11}$$

$$\begin{bmatrix} \gamma_i\pi_i & \pi_i\eta_i^\top B^\top & \pi_i \\ \pi_i B\eta_i & S & 0 \\ \pi_i & 0 & (1 + \gamma_i)^{-1}r_i^{-1} \end{bmatrix} \succeq 0 \tag{12}$$

for $i = 1, \dots, m$, where we have defined the matrix $\Omega$ which satisfies $Q = \Omega^\top \Omega$.

*Proof:*

i) *Sufficiency*: Suppose that there exist matrices $X \succ 0$, $\Lambda_0 \succ 0$ and $F$ such that inequalities (9) and (10) hold. Then, it follows from Theorem 6.4.1 in [17] that for $\Lambda_0 = \text{diag}\{\lambda_{01}, \dots, \lambda_{0m}\}$, there exists $F_0$ and a sufficiently small $\delta > 0$ such that for all $i = 1, \dots, m$,

$$\begin{aligned} J_i &:= \left\| \begin{bmatrix} \Omega \\ \Lambda^{\frac{1}{2}} F_0 \end{bmatrix} (zI - A - BF_0)^{-1} B\eta_i \right\|_2^2 \\ &< \gamma_i\lambda_{0i} - (1 + \gamma_i)r_i - \delta. \end{aligned} \tag{13}$$

We can design the optimal state feedback gain $F_{\Lambda_0}$ to minimize $J_i$ as follows:

$$F_{\Lambda_0} = -(\Lambda_0 + B^\top X_0 B)^{-1}B^\top X_0 A,$$

where $X_0$ is the stabilizing solution of the ARE

$$X_0 = A^\top X_0 A - A^\top X_0 B(\Lambda_0 + B^\top X_0 B)^{-1}B^\top X_0 A + Q.$$

It is known that the optimal state feedback gain $F_{\Lambda_0}$ stabilizes the system and simultaneously minimizes each cost function $J_i$, with optimal costs given by

$$\min_{F_0 \text{ s.t. } A+BF_0 \text{ stable}} J_i = \eta_i^\top B^\top X_0 B\eta_i, \quad i = 1, \dots, m. \tag{14}$$

The expressions in (13) and (14) lead to

$$\eta_i^\top B^\top X_0 B\eta_i < \gamma_i\lambda_{0i} - (1 + \gamma_i)r_i - \delta, \quad i = 1, \dots, m. \tag{15}$$

Now, let $\Lambda_1 = \text{diag}\{\lambda_{11}, \dots, \lambda_{1m}\}$, with

$$\lambda_{1i} = \frac{1}{\gamma_i}(\eta_i^\top B^\top X_0 B\eta_i + (1+\gamma_i)r_i + \delta), \quad i = 1, \dots, m.$$

By (15), we have $\lambda_{1i} < \lambda_{0i}$ for all $i = 1, \dots, m$. Therefore, $0 \prec \Lambda_1 \prec \Lambda_0$. It follows from Theorem 13.3.1 in [6] that the ARE

$$X_1 = A^\top X_1 A - A^\top X_1 B(\Lambda_1 + B^\top X_1 B)^{-1}B^\top X_1 A + Q$$

has stabilizing solution $X_1 \preceq X_0$. This leads to

$$\eta_i^\top B^\top X_1 B\eta_i \le \gamma_i\lambda_{1i} - (1+\gamma_i)r_i - \delta, \quad i = 1, \dots, m.$$

Subsequently, we can construct $\Lambda_2 \prec \Lambda_1$ with the positive semidefinite solution $X_2$. Repeating the process above, we obtain a sequence $\{\Lambda_0, \Lambda_1, \Lambda_2, \dots\}$ satisfying $\Lambda_0 \succ \Lambda_1 \succ \Lambda_2 \succ \dots \succ \Gamma^{-1}(R + \Gamma R + \delta I)$. The sequence of positive definite solutions $\{X_0, X_1, X_2, \dots\}$ satisfies $X_0 \succeq X_1 \succeq X_2 \succeq \dots \succ 0$ and therefore both sequences converge to $\bar{\Lambda}$ and $\bar{X}$ respectively. These matrices satisfy

$$\eta_i^\top B^\top \bar{X} B\eta_i < \gamma_i\bar{\lambda}_i - (1+\gamma_i)r_i, \quad i = 1, \dots, m,$$
$$\bar{X} = A^\top \bar{X} A - A^\top \bar{X} B(\bar{\Lambda} + B^\top \bar{X} B)^{-1} B^\top \bar{X} A + Q,$$

which correspond to the conditions obtained in Lemma 3.1.

*Necessity*: From Lemma 3.1, suppose that there exist $X \succ 0$ and $\Lambda \succ 0$ such that (4) and (5) hold. Hence, it holds for some $\varepsilon > 0$ and some $F \in \mathbb{R}^{m\times n}$ that, for every $i = 1, \dots, m$,

$$\left\| \begin{bmatrix} \Omega \\ \varepsilon I \\ \Lambda^{\frac{1}{2}} F \end{bmatrix} (zI - A - BF)^{-1} B\eta_i \right\|_2^2 < \gamma_i w_i - (1+\gamma_i)r_i.$$

The term on the left side is minimized simultaneously for every $i = 1, \dots, m$ by

$$F_\varepsilon = -(\Lambda + B^\top X_\varepsilon B)^{-1} B^\top X_\varepsilon A, \tag{16}$$

where $X_\varepsilon$ is the stabilizing solution of the ARE

$$\begin{aligned} X_\varepsilon &= A^\top X_\varepsilon A \\ &\quad - A^\top X_\varepsilon B(\Lambda_0 + B^\top X_\varepsilon B)^{-1} B^\top X_\varepsilon A + \varepsilon^2 I + Q. \end{aligned} \tag{17}$$

It is clear that for every $i = 1, \dots, m$,

$$\begin{aligned} \eta_i^\top B^\top X_\varepsilon B\eta_i &= \left\| \begin{bmatrix} \Omega \\ \varepsilon I \\ \Lambda^{\frac{1}{2}} F_\varepsilon \end{bmatrix} (zI - A - BF_\varepsilon)^{-1} B\eta_i \right\|_2^2 \\ &< \gamma_i\lambda_i - (1+\gamma_i)r_i. \end{aligned}$$

Furthermore, using (16) and (17), we obtain

$$\begin{aligned} X_\varepsilon &= A^\top X_\varepsilon A \\ &\quad - A^\top X_\varepsilon B(\Lambda_0 + B^\top X_\varepsilon B)^{-1} B^\top X_\varepsilon A + \varepsilon^2 I + Q \\ &= (A + BF_\varepsilon)^\top X_\varepsilon (A + BF_\varepsilon) + F_\varepsilon^\top \Lambda F_\varepsilon + \varepsilon^2 I + Q. \end{aligned}$$

So, we obtain the following inequality

$$X_\varepsilon > (A + BF_\varepsilon)^\top X_\varepsilon (A + BF_\varepsilon) + F_\varepsilon^\top \Lambda F_\varepsilon + Q.$$

The inequality (9) follows from ignoring the subscript $\varepsilon$ in the previous inequality.

ii) Suppose that (9) and (10) hold for some $X, \Lambda$ and $F$. We can apply the Schur complement lemma to these inequalities to obtain

$$\begin{bmatrix} X & (A+BF)^\top & F^\top & \Omega^\top \\ A+BF & X^{-1} & 0 & 0 \\ F & 0 & \Lambda^{-1} & 0 \\ \Omega & 0 & 0 & I \end{bmatrix} \succ 0, \tag{18}$$

$$\begin{bmatrix} \gamma_i\lambda_i^{-1} & \lambda_i^{-1}\eta_i^\top B^\top & \lambda_i^{-1} \\ \lambda_i^{-1} B\eta_i & X^{-1} & 0 \\ \lambda_i^{-1} & 0 & (1+\gamma_i)^{-1} r_i^{-1} \end{bmatrix} \succeq 0$$

for $i = 1, \dots, m$. Letting $S := X^{-1}$, $Y := FX^{-1}$, $\Pi := \Lambda^{-1}$ and pre- and post- multiplying the matrix $\mathrm{diag}(S, I, I, I)$ on both sides of (18), we obtain the inequalities (11) and (12).

□

That is, if (11) and (12) are feasible with $S \succ 0$, then the MARE (3) has a stabilizing solution. This feasibility problem is easy to compute by any semidefinite programming solver as, for example, *CVX*, permitting us to numerically define a region, depending on $\Gamma$, in which a stabilizing solution of the MARE in (3) exists.

## IV. Case with $W = 1_{m\times m}/\gamma$

In this case, the MARE in (1) becomes

$$X = A^\top XA - \gamma A^\top XB(B^\top XB + R)^{-1} B^\top XA + Q, \tag{19}$$

for $R \succ 0$, $Q \succeq 0$ and where $\gamma \in (0, 1)$. As stated before, it is known that there exists a critical value for $\gamma$, namely $\gamma_c$, above which a stabilizable solution exists (otherwise it does not). In [1] it is established that the critical value satisfies

$$1 - \frac{1}{\max_i |\lambda_i^u(A)|^2} \le \gamma_c \le 1 - \frac{1}{\prod_i |\lambda_i^u(A)|^2}$$

where $\lambda_i^u(A)$ denotes the unstable eigenvalues of $A$. The extreme cases are known to be achieved when $B$ is square and invertible for the lower bound, and when $B$ is rank 1 for the upper bound.

### A. MARE as a standard ARE subject to constraints

For this type of MARE we also found a connection with a standard ARE with constraints, as stated in Lemma 4.1. For simplicity, we shall use $\delta := 1/\gamma$.

*Lemma 4.1:* Assume that the pair $(A, B)$ is controllable. The MARE (19) has a stabilizing solution if and only if there exist positive definite matrices $\Lambda$ and $X$ such that

$$\Lambda - \delta R - (\delta - 1)B^\top XB \succeq 0, \tag{20}$$

where $X$ is a stabilizing solution of the DARE

$$X = A^\top XA - A^\top XB(B^\top XB + \Lambda)^{-1} B^\top XA + Q. \tag{21}$$

*Proof:* If (19) holds for some $X \succ 0$, take

$$\Lambda = \delta R + (\delta - 1)B^\top XB.$$

We see that $\Lambda \succ 0$, and the conditions (20) and (21) are satisfied for these matrices. For the converse, we assume that the pair $(X_0, \Lambda_0)$ satisfies (20) and (21). We can construct a matrix $\Lambda_1$ as

$$\Lambda_1 = (\delta - 1)B^\top X_0 B + \delta R. \tag{22}$$

Clearly, $0 \prec \Lambda_1 \preceq \Lambda_0$. Since the pair $(A, B)$ is controllable, it follows from Theorem 13.3.1 in [6] that the DARE

$$X_1 = A^\top X_1 A - A^\top X_1 B(B^\top X_1 B + \Lambda_1)^{-1} B^\top X A + Q.$$

has stabilizing solution $X_1 \preceq X_0$. Subsequently, we can construct $\Lambda_2 \preceq \Lambda_1$ by (22) with $X_1$ instead of $X_0$. Repeating the process above, we obtain a sequence $\{\Lambda_0, \Lambda_1, \Lambda_2, \dots\}$ satisfying $\Lambda_0 \succeq \Lambda_1 \succeq \Lambda_2 \succeq \cdots \succ 0$. The sequence of positive definite solutions $\{X_0, X_1, X_2, \dots\}$ satisfies $X_0 \succeq X_1 \succeq X_2 \succeq \cdots \succ 0$ and therefore both sequences converge to $\bar{\Lambda}$ and $\bar{X}$ respectively. These matrices satisfy

$$\begin{cases} \bar{X} = A^\top \bar{X} A - A^\top \bar{X} B(B^\top \bar{X} B + \bar{\Lambda})^{-1} B^\top \bar{X} A + Q, \\ \bar{\Lambda} = (\delta - 1)B^\top \bar{X} B + \delta R, \\ \bar{X} = \bar{X}^\top \succ 0, \end{cases}$$

which concludes the proof. □

Lemma 4.1 is analogous to Lemma 3.1 in the previous section but, for the MARE in (19), the constraint is on the whole matrix $B^\top X B$ rather than its diagonal elements. Nevertheless, we can still analyze the existence of a stabilizing solution within a stochastic control framework.

### B. Conditions in terms of linear matrix inequalities

Now we show that the existence of a mean square stabilizing solution for this MARE can be checked by solving a feasibility problem of LMIs.

*Theorem 4.1:* Assume that the pair $(A, B)$ is controllable, and write $Q = \Omega^\top \Omega$. The MARE in (19) has a positive definite and stabilizing solution if and only if there exist $S \succ 0$, $\Pi \succ 0$ and $Y$ such that

$$\begin{bmatrix} S & SA^\top + Y^\top B^\top & Y^\top & S\Omega^\top \\ AS + BY & S & 0 & 0 \\ Y & 0 & \Pi & 0 \\ \Omega S & 0 & 0 & I \end{bmatrix} \succ 0, \tag{23}$$

$$\begin{bmatrix} \Pi & \sqrt{\delta - 1}\Pi B^\top & \sqrt{\delta}\Pi \\ \sqrt{\delta - 1}B\Pi & S & 0 \\ \sqrt{\delta}\Pi & 0 & R^{-1} \end{bmatrix} \succeq 0 \tag{24}$$

*Proof:* First we show that the necessary and sufficient conditions obtained in Lemma 4.1 are equivalent to the existence of matrices $X \succ 0$, $F$ and $\Lambda$ such that

$$X \succeq (A + BF)^\top X(A + BF) + F^\top \Lambda F + Q,$$
$$\Lambda \succeq \delta R + (\delta - 1)B^\top X B.$$

For sufficiency, note that since $R \succ 0$, we have $\Lambda \succ 0$ and therefore we can consider a particular choice for $F$ as

$$F = -(\Lambda + B^\top X B)^{-1} B^\top X A. \tag{25}$$

With this election, we reach the DARE (21). Conversely, if we have the conditions in Lemma 4.1, then we can readily write the DARE as the equality

$$X = (A + BF)^\top X(A + BF) + F^\top \Lambda F + Q,$$

where $F$ is as chosen in (25).

Knowing these alternative expressions, we can mimic the procedure in the proof of Theorem 3.1 to obtain the equivalent conditions:

$$\begin{bmatrix} X & (A + BF)^\top & F^\top & \Omega^\top \\ (A + BF) & X^{-1} & 0 & 0 \\ F & 0 & \Lambda^{-1} & 0 \\ \Omega & 0 & 0 & I \end{bmatrix} \succ 0, \tag{26}$$

$$\begin{bmatrix} \Lambda & \sqrt{\delta - 1}B^\top & \sqrt{\delta}I \\ \sqrt{\delta - 1}B & X^{-1} & 0 \\ \sqrt{\delta}I & 0 & R^{-1} \end{bmatrix} \succeq 0. \tag{27}$$

Letting $S := X^{-1}$, $Y := FX^{-1}$, $\Pi := \Lambda^{-1}$ and pre- and post-multiplying by the matrices $\mathrm{diag}(S, I, I, I)$ and $\mathrm{diag}(\Pi, I, I)$ on both sides of (26) and (27) respectively, we obtain the inequalities (23) and (24). □

## V. EXAMPLES

In the previous sections, we have shown that a wide class of MAREs can be analyzed by checking feasibility of certain LMIs. Here, we give examples where these derivations can be applied.

For the case when $W = 1_{m\times m} + \Gamma^{-1}$ we consider a feedback control problem in which the plant-controller communication is made through signal-to-noise ratio (SNR) constrained channels [18]. Conditions for the existence of stabilizing controllers in such framework is related to the existence of a stabilizing solution of the MARE in (3). Thus, we can replicate the region of SNRs that yields stabilizability given in [18] using our results. For this scenario, $\Gamma$ contains the SNR upper bound of each individual communication channel, the matrices $A$ and $B$ correspond to the state-space matrices of the linear and time-invariant system, and the matrices $Q$ and $R$ are set to $Q = 0$ and $R = 0.001I_{2\times 2}$.

We study the existence of a stabilizing solution of three MAREs, corresponding to the following system matrices:

$$A_1 = \mathrm{diag}(2, 3, 4), \quad B_1 = \begin{bmatrix} 5 & 2 & 0 \\ 0 & 4 & 3 \end{bmatrix}^\top,$$

$$A_2 = \mathrm{diag}(2, 3, 1.5, 4), \quad B_2 = \begin{bmatrix} 5 & 1 & 6 & 0 \\ 0 & 8 & 9 & 3 \end{bmatrix}^\top,$$

$$A_3 = \mathrm{diag}(2, 3, 1.5, 4), \quad B_3 = \begin{bmatrix} 5 & 9 & 8 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}^\top.$$

In Figure 1, we have plotted the limiting values $(\Gamma_1, \Gamma_2)$ that yield stabilizing solutions of the MARE (3) for the three different systems given in [18]. More precisely, each SNR pair $(\Gamma_1, \Gamma_2)$ that is above certain curve, permits a stabilizing solution of the MARE. We can see that the SNR limits naturally depend on the directions associated with each unstable pole.

We also studied the case when $W = 1_{m\times m}/\gamma$ based on the application given in [1]. Specifically, we use the model of the *pendubot*, which consist of a two-link planar robot that must be stabilized in the up-right position. A Kalman filtering based controller is designed for that task in [1] assuming random measurement loss. In such scenario, the existence of such filter

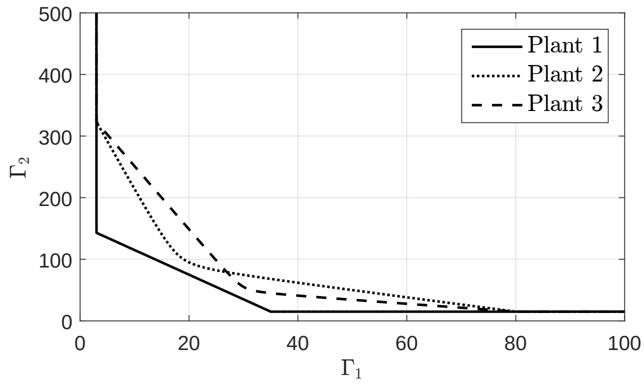


Fig. 1: Lower boundaries of the region of matrices $\Gamma$ that allow a stabilizing solution of the MARE (3).

is linked to the existence of a stabilizing solution of a MARE of the form in (19), where $\gamma$ corresponds to the probability of receiving the measure successfully.

From the model of the pendubot in [1] we have that[1]

$$A = \begin{bmatrix} 0.941 & 0 & 0 & 0 \\ 0 & 0.968 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix}, \quad B = \begin{bmatrix} 0.030 & -0.083 \\ -0.160 & -0.156 \\ 0.146 & 0.149 \\ -0.029 & 0.083 \end{bmatrix},$$

where $(\alpha, \beta)$ are parameters originally set to $(1.033, 1.061)$ in [1]. We also use $R = 0.001\, I_{2\times 2}$ and $Q = qq^\top$, with $q = [0.003 \;\; 1 \;\; -0.005 \;\; -2.15]^\top$. The goal is to find a lower limit over which a stabilizing solution of the MARE (19) exists when varying $\alpha$ and $\beta$. We plotted such surface in Figure 2, where it is clear that the requirement on $\gamma$ increases as the unstable poles locations move far from the unit circle.

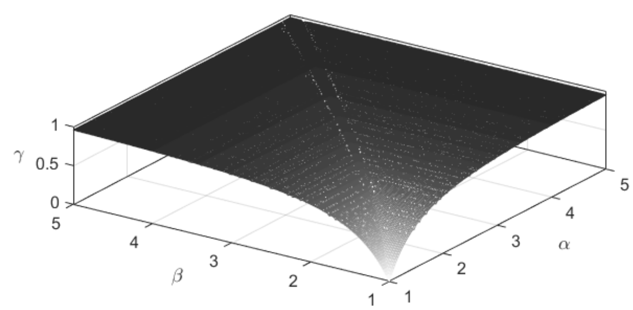


Fig. 2: Lower limit of $\gamma$ that allows a stabilizing solution of the MARE (19), for different pole locations.

## VI. Conclusions

In this paper we have studied two types of modified algebraic Riccati equations and have related the existence of stabilizing solutions with alternative problems. One of these alternative problems arises when rewriting the MARE as a standard ARE whose solution is subject to a specific constraint. This permitted us to view the MARE-related stabilization problems from a different perspective and use well known results and properties on AREs to rewrite the existence of stabilizing solutions considering the aforementioned constraints. The second alternative problem corresponded to an LMI problem whose feasibility is directly related to the existence of a stabilizing solution of the original MARE.

[1]Notice that we have used the modal state representation of the pendubot, and also have changed $C \to B^\top$ in order to use the dual MARE.

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