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Refined instrumental variable methods for unstable continuous-time systems in closed-loop

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ABSTRACT
In continuous-time system identification, refined instrumental variable methods are widely used in open and closed-loop settings. Although their robustness and performance are well documented for stable systems, these estimators are not reliable for estimating unstable continuous-time models. The main difficulty we encounter in modeling unstable systems with refined instrumental variables is that the filtered regressor and instrument vectors, as well as the filtered output, become severely ill-conditioned if the model is unstable during the iterative process. In this work, we propose a solution to this problem by including a tailor-made all-pass filter in the prefiltering step. This approach is used for obtaining an extension of the least-squares state-variable filter method, as well as extensions for the refined instrumental variable method for continuous-time systems (RIVC) and its simplified embodiment (SRIVC), that admit the identification of unstable systems and are shown to minimize the prediction error upon convergence and as the sample size goes to infinity. In addition, several implementations of these methods are proposed depending on the intersample behavior of the input (zero and first-order hold, multisine and arbitrary). The particular case when the plant has integral action is explicitly considered in this work. In an indirect system identification setting, an extension of the closed-loop version of the SRIVC method is also proposed and discussed in detail. Monte Carlo simulations are used to assess the performance of our methods.

KEYWORDS
System identification; continuous-time systems; instrumental variables; unstable systems

1. Introduction

The greatest difficulty surrounding the estimation of unstable linear and time-invariant models is that most system identification algorithms, directly or indirectly, use predictors or regressors that become unstable during the identification process. An illustrating example of this problem is encountered in the Simplified Refined Instrumental Variable method for Continuous-time systems (SRIVC, Young and Jakeman (1980)) when input and output data are used for identification. This estimator has been proven to be generically consistent (Pan, González, Welsh, & Rojas, 2020) and asymptotically efficient (Pan, Welsh, González, & Rojas, 2020b) under mild condi-

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tions, including the assumption that the true system and the model at every iteration of the algorithm are asymptotically stable. The stability of the model iterates is needed for the filtering process that must be performed: since the input and output signals are prefiltered by auxiliary estimates of the true system, such a procedure becomes severely ill-conditioned if these estimates are unstable. In this work, we are concerned about modeling unstable continuous-time systems in a closed-loop setting.

Closed-loop system identification has mostly been studied in the discrete-time domain (Forssell & Ljung, 1999; Gilson, Garnier, Young, & Van den Hof, 2011; Van den Hof, 1998). Nevertheless, directly identifying continuous-time systems in closed-loop is known to have several advantages over discrete-time identification, such as a greater physical insight on the system parameters and controller tuning, ease of incorporating non-uniform and fast sampling, and more flexibility in the proposed model structure (Garnier & Young, 2014). In closed-loop continuous-time system identification, different algorithms are suggested according to whether the controller is known or not. If only input and output data are known, then one approach to direct closed-loop continuous-time system identification is to apply the RIVC or SRIVC methods (Young & Jakeman, 1980). If the controller is not known but reference signal data are available, Young, Garnier, and Gilson (2009) have provided two two-stage algorithms that are suggested to be consistent but not asymptotically efficient, and convergence is not guaranteed for unstable plants (Young, 2012, Page 274, Chapter 9). A three-stage method has also been proposed in Young et al. (2009), but it is not applicable when the plant is unstable. On the other hand, if the controller is linear and time-invariant and known a priori, then an attractive approach is to apply the CLRIVC or CLSRIVC method as proposed in Chapter 5 of Garnier and Wang (2008). In such case, the idea is to formulate the instrument vector by prefiltering the reference signal with estimates obtained in the previous iteration of the closed-loop transfer function multiplied by the open-loop denominator transfer function. Again, these methods are not suitable for identifying unstable systems.

Several extensions of closed-loop methods suffer from the same issue. Ni, Gilson, and Garnier (2012) developed an instrumental variable-based method for estimating Hammerstein-Wiener models, while Chen, Gilson, Agüero, Garnier, and Schorsch (2014) adapted the CLRIVC method to estimate continuous-time systems with non-uniformly sampled data. Victor, Diudichi, and Melchior (2017) proposed using instrumental variables for closed-loop identification in an errors-in-variables framework, and Padilla, Garnier, Young, and Yuz (2017) explored recursive identification. Recently, González, Rojas, Pan, and Welsh (2021) extended the applicability of both SRIVC and CLSRIVC estimators for arbitrary inputs, and these estimators were shown to lose their consistency if the intersample behavior of the system input is not taken into account. All the methods mentioned above are based on refined instrumental variables and thus cannot be directly applied if the open-loop system is unstable.

In contrast to the aforementioned work, the main contributions of this work are the following:

- We present an extension of the least-squares state-variable filter (LSSVF) method that can serve as an initialization step for any subsequent identification method for unstable continuous-time systems.
- We propose an extension to the SRIVC and RIVC methods for estimating continuous-time unstable systems in Box-Jenkins (BJ) and output error (OE) model structures, respectively. These refinements admit zero and first-order hold (ZOH and FOH) input signals, as well as continuous-time multisines and arbi-

trary inputs whose intersample behaviors are known.

- We discuss how to handle unstable systems for indirect closed-loop continuous-time system identification using the CLSRIVC method.
- We provide variants of the SRIVC, RIVC and CLSRIVC estimators that can deal with plants with integral action. This type of system is a limit case that is not adequately covered in the extensions previously proposed.
- We test the proposed methods via extensive numerical simulations.

The rest of this work is organized as follows. In Section 2 we present the framework of our work and state the identification problem. Section 3 introduces the RIVC and SRIVC estimators, while Section 4 contains our main contribution, namely, the derivation of the RIVC and SRIVC estimators that admit unstable continuous-time models. Section 5 studies the applicability of our approach to indirect closed-loop continuous-time system identification methods, and Section 6 covers identification with integral action in the plant. In Section 7, extensive simulations are performed in order to test the proposed estimators under different experimental conditions, and we provide concluding comments in Section 8. Proofs of the theoretical results of this article can be found in the appendix.

Notation: The forward-shift operator is denoted by q and satisfies $qf(t_k) = f(t_{k+1})$, while the Heaviside operator is denoted by p and satisfies $pg(t) = \mathrm{d}g(t)/\mathrm{d}t$. The symbol $\{G(p)x(t)\}_{t=t_k}$ (or $[G(p)x(t)]_{t=t_k}$ in the vector-valued case) means that the continuous-time signal $x(t)$ is filtered through the continuous-time transfer function $G(p)$ and later sampled at $t = t_k$. On the other hand, the notation $G(p)x(t_k)$ implies that the signal $x(t_k)$ is interpolated using either a ZOH or FOH, and the resulting output of the filter is evaluated at t_k. The Delta-domain equivalent of $G(p) = B(p)/A(p)$ is denoted as $G_\Delta(\delta) = B_\Delta(\delta)/A_\Delta(\delta)$ and is a function of the delta operator δ.

2. Problem formulation

We begin with an analysis of the direct closed-loop system identification problem. Consider the proper, linear and time-invariant continuous-time system

$$x(t) = \frac{B^*(p)}{A^*(p)}u(t).$$

The system numerator and denominator polynomials are assumed to be coprime with orders m^* and n^*, respectively, i.e.,

$$\begin{aligned} B^*(p) &= b^*_{m^*}p^{m^*} + b^*_{m^*-1}p^{m^*-1} + \cdots + b^*_1 p + b^*_0, \\ A^*(p) &= a^*_{n^*}p^{n^*} + a^*_{n^*-1}p^{n^*-1} + \cdots + a^*_1 p + 1, \end{aligned} \tag{1}$$

where $m^* \leq n^*$, and the system parameter vector is described by

$$\boldsymbol{\theta}^* := \begin{bmatrix} a^*_1, & a^*_2, & \ldots, & a^*_{n^*}, & b^*_0, & b^*_1, & \ldots, & b^*_{m^*} \end{bmatrix}. \tag{2}$$

A noisy measurement of the sampled output is retrieved for identification:

$$y(t_k) = x(t_k) + \frac{C^*(q)}{D^*(q)}e(t_k),$$

where $e(t_k)$ describes a zero-mean white noise stochastic process of finite variance, and

$$C^*(q) = 1 + c_1^* q^{-1} + c_2^* q^{-2} + \cdots + c_{m_c^*}^* q^{-m_c^*},$$
$$D^*(q) = 1 + d_1^* q^{-1} + d_2^* q^{-2} + \cdots + d_{n_d^*}^* q^{-n_d^*},$$

where $n_d^* \geq m_c^*$. The parameters in the noise model can be described by the vector

$$\boldsymbol{\eta}^* := \begin{bmatrix} c_1^*, & c_2^*, & \ldots, & c_{m_c^*}^*, & d_1^*, & d_2^*, & \ldots, & d_{n_d^*}^* \end{bmatrix}. \tag{3}$$

Throughout this work, the transfer function $G^*(p) := B^*(p)/A^*(p)$ is assumed to be *unstable*. That is, at least one zero of the polynomial $A^*(p)$ is in the closed right-half of the complex plane. We also assume that the discrete-time filter $H^*(q) := C^*(q)/D^*(q)$ is stable and minimum-phase. Recall that by the spectral factorization theorem (Papoulis & Pillai, 2002), any colored noise with rational and integrable spectrum can be described by a noise white stochastic process filtered by a stable and minimum-phase transfer function.

Our goal is to estimate the transfer function $G^*(p)$ and the noise model $H^*(q)$ using sampled data. We require N samples of input and output data $\{u(t_k), y(t_k)\}_{k=1}^N$, where the sampling instants $\{t_k\}_{k=1}^N$ are assumed to be evenly spaced in time with a sampling period h. Since unstable systems are usually operated in closed-loop, data are assumed to be generated in a closed-loop that stabilizes the plant. However, at this stage, no information about the closed-loop is assumed. We further assume that the intersample behavior of the input is known but can be arbitrary (i.e., it is not necessarily perfectly described via ZOH or FOH devices). This aspect, along with the nature of the transfer functions, separates this analysis from the standard discrete-time approaches (Forssell & Ljung, 1999), which typically assume that the intersample behavior of the signals has already been addressed prior to the description of the systems.

The model we consider is the following:

$$y(t_k) = \left\{ \frac{B(p)}{A(p)} u(t) \right\}_{t=t_k} + \frac{C(q)}{D(q)} \varepsilon(t_k), \tag{4}$$

where the signal $\varepsilon(t_k)$ is denoted as the residual, and the transfer functions $G(p) = B(p)/A(p)$ and $H(q) = C(q)/D(q)$ are described by polynomials that are parameterized[1] by vectors $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ of the same form as (2) and (3), respectively. For simplicity, we assume that the system and noise models exactly parameterize the true system, i.e., $n = n^*$, $m = m^*$, $m_c = m_c^*$ and $n_d = n_d^*$, although this condition can be relaxed without major setbacks on the description of our algorithms. The first goal is to adapt the SRIVC and RIVC methods to handle unstable systems, and then to propose extensions to these methods to be used in an indirect closed-loop system identification setting that will be described more carefully in Section 5.

3. The RIVC and SRIVC methods

The RIVC estimator is an iterative instrumental variable method for the identification of hybrid BJ models, in which the plant is described by a continuous-time transfer

[1] In some sections of this work, we will write $A(p, \boldsymbol{\theta})$ instead of $A(p)$, or $C(q, \boldsymbol{\eta})$ instead of $C(q)$, to show explicit dependence of the model polynomials on the unknown parameters $\boldsymbol{\theta}$ or $\boldsymbol{\eta}$, respectively.

function and the noise model is discrete in time. Each iteration of the procedure consists in a noise model estimation step, followed by the system parameter estimation step using instruments and regressors that are computed from a bank of prefilters that depend on the current system and noise models.

The RIVC method seeks to minimize a least-squares criterion of the following error function:

$$\varepsilon(t_k) = \frac{D(q)}{C(q)} \left(y(t_k) - \frac{B(p)}{A(p)} u(t_k) \right). \tag{5}$$

In the continuous-time system identification literature, the expression in (5) is called the *generalized equation error* (GEE, Young (1981)), and it can be interpreted as the one-step ahead prediction error for BJ models (Ljung, 1999). The GEE can also be written as

$$\varepsilon(t_k) = y_f(t_k) + a_1 y_f^{(1)}(t_k) + \cdots + a_n y_f^{(n)}(t_k) - b_0 u_f(t_k) - \cdots - b_m u_f^{(m)}(t_k),$$

where

$$y_f^{(i)}(t_k) = \frac{D(q)}{C(q)} \frac{p^i}{A(p)} y(t_k), \quad i = 0, \dots, n; \tag{6a}$$

$$u_f^{(l)}(t_k) = \frac{D(q)}{C(q)} \frac{p^l}{A(p)} u(t_k), \quad l = 0, \dots, m. \tag{6b}$$

Thus, instead of a standard linear regression, we have derived a *pseudo-linear regression*[2] (Solo, 1978) of the form

$$y_f(t_k) = \boldsymbol{\varphi}_f^\top(t_k)\boldsymbol{\theta} + \varepsilon(t_k), \tag{7}$$

where

$$\boldsymbol{\varphi}_f(t_k) = \frac{D(q)}{C(q)} \left[-\frac{p}{A(p)} y(t_k), \dots, -\frac{p^n}{A(p)} y(t_k), \frac{1}{A(p)} u(t_k), \dots, \frac{p^m}{A(p)} u(t_k) \right]^\top. \tag{8}$$

Since $1/A(p)$ and $D(q)/C(q)$ are unknown, as they depend on the unknown parameters, the RIVC method proposes to estimate them via iterations. To this end, assume that the estimate at the j-th iteration, $\boldsymbol{\theta}_j$, is known, with its associated model $G_j(p) := B_j(p)/A_j(p)$. First, an estimate of the noise model $C_{j+1}(q)/D_{j+1}(q)$ with parameter vector $\boldsymbol{\eta}_{j+1}$ is obtained by fitting an ARMA model to the estimated noise sequence $y(t_k) - G_j(p)u(t_k)$. This estimation step can be performed using standard discrete-time methods such as PEM (Ljung, 1999) or IVARMA (Young, 2006). Afterwards, the filtering in (6) is done with the inverse of the noise model previously obtained and continuous-time transfer functions of the form $p^l/A_j(p)$. Lastly, an instrumental

[2] *Pseudo* refers to the fact that the parameter vector $\boldsymbol{\theta}$ is also present in the regressor vector $\boldsymbol{\varphi}_f(t_k)$ and the filtered output $y_f(t_k)$. This dependence on the parameter vector is made explicit in our notation only when necessary.

variable approach is used to form the next iterate of the system parameters:

$$\boldsymbol{\theta}_{j+1} = \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)y_f(t_k)\right], \tag{9}$$

where the filtered instrument vector, $\hat{\boldsymbol{\varphi}}_f(t_k)$, is given by

$$\hat{\boldsymbol{\varphi}}_f(t_k) = \frac{D_{j+1}(q)}{C_{j+1}(q)}\left[-\frac{pB_j(p)}{A_j^2(p)}u(t_k), \dots, -\frac{p^nB_j(p)}{A_j^2(p)}u(t_k), \frac{1}{A_j(p)}u(t_k), \dots, \frac{p^m}{A_j(p)}u(t_k)\right]^\top, \tag{10}$$

and the filtered regressor vector $\boldsymbol{\varphi}_f(t_k)$ is obtained as in (8) with $A_j(p), C_{j+1}(q)$ and $D_{j+1}(q)$ in place of $A(p)$, $C(q)$ and $D(q)$ respectively. If the noise model is not of interest, then the simplified RIVC estimator (SRIVC) can be computed by setting $C_j(q) = D_j(q) = 1$ and computing (9) until convergence.

The initialization of the RIVC and SRIVC procedures is an important step that needs discussion. Most often, a least-squares state-variable filter (LSSVF) method is applied (Young, 1966), which consists in proposing a fixed filter $1/A_1(p)$ and obtaining the least-squares solution

$$\boldsymbol{\theta}_{LSSVF} = \left[\sum_{k=1}^{N} \boldsymbol{\varphi}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \boldsymbol{\varphi}_f(t_k)y_f(t_k)\right],$$

where $C(q)$ and $D(q)$ have been set to one in both (6) and (8). The fixed filter $1/A_1(p)$ should be designed based on prior knowledge about the system, and it is recommended to be set as close to $1/A^*(p)$ as possible (Pan, Welsh, González, & Rojas, 2020a).

One shortcoming that all refined instrumental variable estimators have is that they do not deliver reliable estimates if the plant we are identifying is unstable. This is due to the filtering procedure for generating the filtered output, regressor and instrument vectors, i.e., they require filtering by $1/A_j(p)$, which would be unstable if an unstable transfer function is sought.

Remark 1. Recently, it has been proven that a misspecified intersample behavior of the input signal for the generation of the regressor and instrument vectors affects the consistency and asymptotic efficiency of the SRIVC method (González et al., 2021; Pan, González, et al., 2020; Pan, Welsh, González, & Rojas, 2020b). In this work we will consider the optimal implementation of these methods, that is, we assume that the *exact* continuous-time input is used for constructing the filtered regressor and instrument vectors:

$$\begin{aligned}\boldsymbol{\varphi}_f(t_k) &= \frac{D_{j+1}(q)}{C_{j+1}(q)}\left[\frac{-p}{A_j(p)}y(t_k), \dots, \frac{-p^n}{A_j(p)}y(t_k), \left\{\frac{1}{A_j(p)}u(t)\right\}_{t=t_k}, \dots, \left\{\frac{p^m}{A_j(p)}u(t)\right\}_{t=t_k}\right]^\top, \\ \hat{\boldsymbol{\varphi}}_f(t_k) &= \frac{D_{j+1}(q)}{C_{j+1}(q)}\left[-\frac{pB_j(p)}{A_j^2(p)}u(t), \dots, -\frac{p^nB_j(p)}{A_j^2(p)}u(t), \frac{1}{A_j(p)}u(t), \dots, \frac{p^m}{A_j(p)}u(t)\right]_{t=t_k}^\top.\end{aligned} \tag{11}$$

4. RIVC and SRIVC for unstable systems

In this section, we derive extensions of the RIVC, SRIVC, and LSSVF methods that can identify stable or unstable continuous-time plants. We first study the optimal one-step-ahead predictor for a BJ model structure with unstable continuous-time systems. We then propose the filtered regressor and output vectors to be used for the extension of the RIVC and SRIVC methods, and present an adequate initialization procedure based on the LSSVF method. Finally, we study the instrument vector constructed such that a Prediction Error Method (PEM) estimate is obtained upon convergence.

4.1. One-step-ahead predictor and a refined least-squares method

The RIVC algorithm implicitly uses the predictor

$$\hat{y}(t_k|\boldsymbol{\theta}) = \frac{D(q)}{C(q)}\left\{\frac{B(p)}{A(p)}u(t)\right\}_{t=t_k} + \left(1-\frac{D(q)}{C(q)}\right)y(t_k).$$

If we encounter an unstable open-loop system, this predictor is not feasible for computations within the RIVC paradigm. Thus, our first goal is to extend the applicability of this estimator to the unstable system case. This extension must admit ZOH and FOH input signals, as well as arbitrary input signals whose intersample behavior is known, such as in the case of continuous-time multisines or band-limited signals (González, Rojas, Pan, & Welsh, 2021; González et al., 2021).

Before we derive the stable predictor, we define some polynomials of interest. The denominator of the discrete-time equivalent of $1/A(p)$ is denoted as $A_d(q)$, and it is assumed to be of the form

$$A_d(q) = 1 + a_1^d q^{-1} + \cdots + a_n^d q^{-n}.$$

This polynomial can be written in terms of the coefficients of $A(p)$ via the following relation:

$$1 + a_1^d q^{-1} + \cdots + a_n^d q^{-n} = q^{-n}\det\big(q\mathbf{I} - \exp(\mathbf{A}h)\big), \tag{12}$$

where

$$\mathbf{A} = \begin{bmatrix} -\frac{a_{n-1}}{a_n} & \cdots & -\frac{a_1}{a_n} & -\frac{1}{a_n} \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}. \tag{13}$$

We also consider the decomposition $A_d(q) = A_{d,s}(q)A_{d,a}(q)$, where $A_{d,s}(q)$ denotes the stable part of $A_d(q)$ and $A_{d,a}(q)$ its antistable part. In particular, we write

$$A_{d,s}(q) = 1 + a_1^{d,s} q^{-1} + \cdots + a_{n_s}^{d,s} q^{-n_s},$$
$$A_{d,a}(q) = 1 + a_1^{d,a} q^{-1} + \cdots + a_{n_a}^{d,a} q^{-n_a},$$

where $n_s + n_a = n$ and $a_{n_a}^{d,a}$ is assumed to be different from zero without loss of generality. Note that with the notation

$$\tilde{a}_k^{d,s} := \begin{cases} 1, & k=0 \\ a_k^{d,s}, & 1 \leq k \leq n_s \\ 0, & \text{otherwise;} \end{cases} \quad \tilde{a}_k^{d,a} := \begin{cases} 1, & k=0 \\ a_k^{d,a}, & 1 \leq k \leq n_a \\ 0, & \text{otherwise,} \end{cases}$$

the following relation holds:

$$a_k^d = \sum_{j=0}^{n} \tilde{a}_j^{d,s} \tilde{a}_{k-j}^{d,a}. \tag{14}$$

Furthermore, we denote by $\overline{A}_{d,a}(q)$ the monic polynomial whose zeros are those of $A_{d,a}(q)$ reflected into the unit disc:

$$\overline{A}_{d,a}(q) = 1 + \frac{a_{n_a-1}^{d,a}}{a_{n_a}^{d,a}} q^{-1} + \cdots + \frac{a_1^{d,a}}{a_{n_a}^{d,a}} q^{-n_a+1} + \frac{1}{a_{n_a}^{d,a}} q^{-n_a}.$$

Our approach relies on studying the following model structure

$$y(t_k) = \left\{ \frac{B(p)}{A(p)} u(t) \right\}_{t=t_k} + \frac{\tilde{C}(q)}{\tilde{D}(q)} \varepsilon(t_k), \tag{15}$$

where

$$\tilde{C}(q) = C(q)\overline{A}_{d,a}(q), \text{ and } \tilde{D}(q) = D(q)A_{d,a}(q). \tag{16}$$

A model structure with a tailor-made noise model as in (15) was introduced for discrete-time system identification in Forssell and Ljung (2000), and it has been used for identifying unstable systems with the Weighted Null-Space Fitting method (Galrinho, Rojas, & Hjalmarsson, 2016), as well as for ARX modeling (Galrinho, Everitt, & Hjalmarsson, 2017). To the best of our knowledge, this approach has not been applied in continuous-time system identification until now.

With the previous definitions in mind, we are ready to derive a stable predictor and its properties for this case. In the sequel, we will explicitly write the dependence of the polynomials and vectors on the parameter vectors $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$.

Proposition 4.1. *Consider the model structure* (15)*, where* $\varepsilon(t_k)$ *is assumed to be white noise. The optimal one-step-ahead predictor is given by*

$$\hat{y}(t_k|\boldsymbol{\theta},\boldsymbol{\eta}) = \frac{\tilde{D}(q,\boldsymbol{\eta},\boldsymbol{\theta})}{\tilde{C}(q,\boldsymbol{\eta},\boldsymbol{\theta})} \left\{ \frac{B(p,\boldsymbol{\theta})}{A(p,\boldsymbol{\theta})} u(t) \right\}_{t=t_k} + \left(1 - \frac{\tilde{D}(q,\boldsymbol{\eta},\boldsymbol{\theta})}{\tilde{C}(q,\boldsymbol{\eta},\boldsymbol{\theta})} \right) y(t_k). \tag{17}$$

Proof. See Appendix A. □

The difference between Proposition 4.1 and the standard result for discrete-time models (see, e.g., Chapter 3 of Ljung (1999)) is that the plant is a continuous-time transfer function. Thus, the cascading of discrete-time and continuous-time transfer functions must be done with caution: the model output must be treated as a discrete-time signal instead of a continuous-time one. Nevertheless, the proof follows the same lines as its discrete-time version.

Similar to the discrete-time case, minimizing the 2-norm of the prediction error of the model structure in (15) can be shown to be asymptotically equivalent to minimizing the 2-norm of the prediction error for the model structure (4).

Proposition 4.2. *When applying the prediction error method to model structures* (4) *and* (15)*, the resulting estimates of the two model structures will asymptotically (as* $N \to \infty$*) be the same.*

Proof. The proof follows the same lines as the proof of Proposition 1 of Forssell and Ljung (2000). □

Proposition 4.1 provides a path to writing a modification of the GEE that has stable filters, which is essential for computing the RIVC estimate. We write the residual $\varepsilon(t_k)$ as

$$\varepsilon(t_k) = y(t_k) - \hat{y}(t_k|\boldsymbol{\theta}, \boldsymbol{\eta}) \tag{18}$$

$$= \frac{\tilde{D}(q, \boldsymbol{\eta}, \boldsymbol{\theta})}{\tilde{C}(q, \boldsymbol{\eta}, \boldsymbol{\theta})} A(p, \boldsymbol{\theta}) \frac{1}{A(p, \boldsymbol{\theta})} y(t_k) - \frac{\tilde{D}(q, \boldsymbol{\eta}, \boldsymbol{\theta})}{\tilde{C}(q, \boldsymbol{\eta}, \boldsymbol{\theta})} \left\{ \frac{B(p, \boldsymbol{\theta})}{A(p, \boldsymbol{\theta})} u(t) \right\}_{t=t_k}. \tag{19}$$

Thus, the regression equation can now be written as

$$y_{f,uns}(t_k) = \boldsymbol{\varphi}^\top_{f,uns}(t_k)\boldsymbol{\theta} + \varepsilon(t_k), \tag{20}$$

where

$$y_{f,uns}(t_k) = \frac{\tilde{D}(q, \boldsymbol{\eta}, \boldsymbol{\theta})}{\tilde{C}(q, \boldsymbol{\eta}, \boldsymbol{\theta})} \frac{1}{A(p, \boldsymbol{\theta})} y(t_k), \tag{21}$$

$$\boldsymbol{\varphi}^\top_{f,uns}(t_k) = \frac{\tilde{D}(q,\boldsymbol{\eta},\boldsymbol{\theta})}{\tilde{C}(q,\boldsymbol{\eta},\boldsymbol{\theta})} \left[\frac{-p}{A(p,\boldsymbol{\theta})} y(t_k), \dots, \frac{-p^n}{A(p,\boldsymbol{\theta})} y(t_k), \left\{ \frac{1}{A(p,\boldsymbol{\theta})} u(t) \right\}_{t=t_k}, \dots, \left\{ \frac{p^m}{A(p,\boldsymbol{\theta})} u(t) \right\}_{t=t_k} \right]. \tag{22}$$

Note that the goal of this description is to render filtered signals that are bounded, despite possible instabilities due to filtering by $1/A(p, \boldsymbol{\theta})$. Before deriving the proposed extension for the RIVC estimator (which requires deriving a tailor-made filtered instrument vector), a refined least-squares method will be introduced. Assuming knowledge of an initial model estimate, the refined least-squares method we study consists of four steps:

(1) Based on the j-th iteration of the system model (i.e., $\boldsymbol{\theta}_j$), fit an ARMA model $v(t_k) = \frac{C(q,\boldsymbol{\eta}_{j+1})}{D(q,\boldsymbol{\eta}_{j+1})} e(t_k)$ to the estimated colored noise sequence

$$\hat{v}(t_k) = \frac{A_{d,a}(q, \boldsymbol{\theta}_j)}{\bar{A}_{d,a}(q, \boldsymbol{\theta}_j)} \left(y(t_k) - \left\{ \frac{B(p, \boldsymbol{\theta}_j)}{A(p, \boldsymbol{\theta}_j)} u(t) \right\}_{t=t_k} \right). \tag{23}$$

(2) Form the filtered output and regressor from (21) and (22), where $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$ are set to $\boldsymbol{\eta}_{j+1}$ and $\boldsymbol{\theta}_j$, respectively.

(3) Compute the estimate

$$\boldsymbol{\theta}_{j+1} = \left[\sum_{k=1}^{N} \boldsymbol{\varphi}_{f,uns}(t_k)\boldsymbol{\varphi}_{f,uns}^{\top}(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \boldsymbol{\varphi}_{f,uns}(t_k) y_{f,uns}(t_k)\right]. \tag{24}$$

(4) Go back to step (1) and repeat until convergence.

Several implementation issues regarding the filtering process are discussed next.

(a) The discrete-time filtering in (23) is needed for stability, since the model estimate will usually be unstable. The unstable poles of the model estimate are canceled with the zeros of $A_{d,a}(q, \boldsymbol{\theta}_j)$, rendering a signal $\hat{v}(t_k)$ that is quasi-stationary if the input is quasi-stationary (Ljung, 1999). Note that the pole-canceling filter only affects the gain of the spectrum of $\hat{v}(t_k)$, since it is designed to be all-pass. Moreover, a byproduct of Proposition 4.2 is that other pole-canceling filters can be proposed, with the same prediction error being minimized asymptotically, as long as the discrete-time filters are all-pass.
(b) Poles of model estimates that are on the imaginary axis are not canceled with our approach. This is not an important setback in a practical setting if the true system does not contain integrators or undamped oscillatory modes, since cases where a pole of the model is found exactly on the imaginary axis occur with probability zero. In Section 6, we study two different approaches for the identification of systems with integral effect that are complementary to the methods presented in this section.
(c) The input signal filtering can be done entirely in discrete-time for ZOH or FOH inputs. In practice, this is done by transforming the continuous-time transfer function into its ZOH or FOH equivalent, multiplying the transfer functions in the q-domain, and later computing the filtered signal via discrete-time filtering. This approach is needed for the cancellation of unstable poles present in $1/A(p, \boldsymbol{\theta}_j)$ with the zeros in $A_{d,a}(q, \boldsymbol{\theta}_j)$. Note that for such types of inputs, discrete-time filtering (i.e., via ZOH or FOH-equivalent transfer functions) is commonly used for computing the regressor and filtered output (Garnier & Wang, 2008).
(d) For continuous-time multisine input signals, it is sufficient to compute the continuous-time filtering in (22) assuming steady state (which also results in a multisine), and later performing the discrete-time filtering by the inverted noise model assuming a ZOH or FOH intersample behavior for the signals $\{[p^l/A(p, \boldsymbol{\theta}_j)]u(t)\}_{t=t_k}$, $l = 0, 1, \ldots, m$. Note that such inputs are rarely encountered if there is noise present in the loop, which is the case for most practical applications.
(e) The unstable poles of the bank of continuous-time filters are not canceled by $A_{d,a}(q, \boldsymbol{\theta}_j)$ when the input is an arbitrary continuous-time signal. To solve this issue, we propose modifying the oversampling technique suggested in González et al. (2021), which obtains accurate estimates with SRIVC and CLSRIVC under arbitrary inputs by oversampling the input by a factor $S \gg 1$ and performing the filtering in the Delta domain (Middleton & Goodwin, 1990). The previously proposed method for solving the instability problem using the noise model in (16) does not work for this case, since the poles of the equivalent fast-sampled Delta description do not cancel with the zeros of $A_{d,a}(q, \boldsymbol{\theta}_j)$. The solution is described in detail in Appendix B.

Remark 2. It can be shown that the estimator presented in (24) is asymptotically biased, and does not minimize the sum of squares of the prediction errors. However, it is useful for understanding the refinement nature of the instrumental variable methods that we introduce in the next subsection, and its use can be justified if there are computational constraints that limit the use of the other methods we study in this article. Furthermore, one iteration of the estimator presented in (24) with $j = 1$, $C_2(q) = D_2(q) = 1$ and a fixed $1/A_1(p)$ polynomial, provides an extension of the LSSVF method that can admit an unstable state variable filter $1/A_1(p)$. We propose this extended LSSVF method as an alternative for initializing refined instrumental variable methods for unstable systems.

In the sequel, we eliminate the asymptotic bias of the refined LS method by constructing the instrument vector that delivers a PEM estimate for this situation, which leads to novel refined instrumental variables methods that admit unstable models.

4.2. Instrument vector

We are interested in estimators that are computed iteratively using instrumental variables:

$$\boldsymbol{\theta}_{j+1} = \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_{f,uns}(t_k)\boldsymbol{\varphi}_{f,uns}^{\top}(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_{f,uns}(t_k) y_{f,uns}(t_k)\right]. \tag{25}$$

Here the filtered instrument vector is denoted by $\hat{\boldsymbol{\varphi}}_{f,uns}(t_k)$, which uses filters that depend on the estimate of the previous iteration, $\boldsymbol{\theta}_j$, and the current noise model estimate, $\boldsymbol{\eta}_{j+1}$.

Lemma 4.3. *Consider the iterative method in* (25) *but where the noise model is fixed, and assume that at the converging point (i.e.,* $\lim_{j\to\infty} \boldsymbol{\theta}_j$*) the matrix*

$$\frac{1}{N}\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_{f,uns}(t_k)\boldsymbol{\varphi}_{f,uns}^{\top}(t_k)$$

is non-singular, where the instrument vector, $\hat{\boldsymbol{\varphi}}_{f,uns}(t_k)$*, is given by*

$$\hat{\boldsymbol{\varphi}}_{f,uns}(t_k) = \frac{\partial}{\partial \boldsymbol{\theta}} \hat{y}(t_k|\boldsymbol{\theta},\boldsymbol{\eta})\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_j}, \tag{26}$$

with $\hat{y}(t_k|\boldsymbol{\theta},\boldsymbol{\eta})$ *as in* (17)*. If the algorithm in* (25) *converges as* $j \to \infty$ *(with* N *fixed and finite), then the converging point minimizes the sum of squares of the residuals.*

Proof. See Appendix C. □

Remark 3. If the model estimates at each iteration are stable, then Lemma 4.3 applied to (7) leads to the RIVC estimator with instrument vector given by (10), or (11) in the arbitrary input case. This supports the findings of Young (2015), who suggests that the RIVC estimator provides a maximum likelihood estimate for hybrid BJ models with filtered Gaussian noise.

We introduce the estimator computed by (25), where $\hat{\boldsymbol{\varphi}}_{f,uns}(t_k)$ is computed from (26), as the extended RIVC estimator for stable and unstable continuous-time systems. Naturally, by fixing the noise model to $C(q) = D(q) = 1$, this procedure computes the extended SRIVC estimator. An implementation diagram for the extended RIVC estimator can be found in Figure 1. Note that the gradient step can also be performed with the residual $\varepsilon(t_k)$ instead of the one-step predictor $\hat{y}(t_k|\boldsymbol{\theta}, \boldsymbol{\eta})$, due to the relation between these signals in (18).

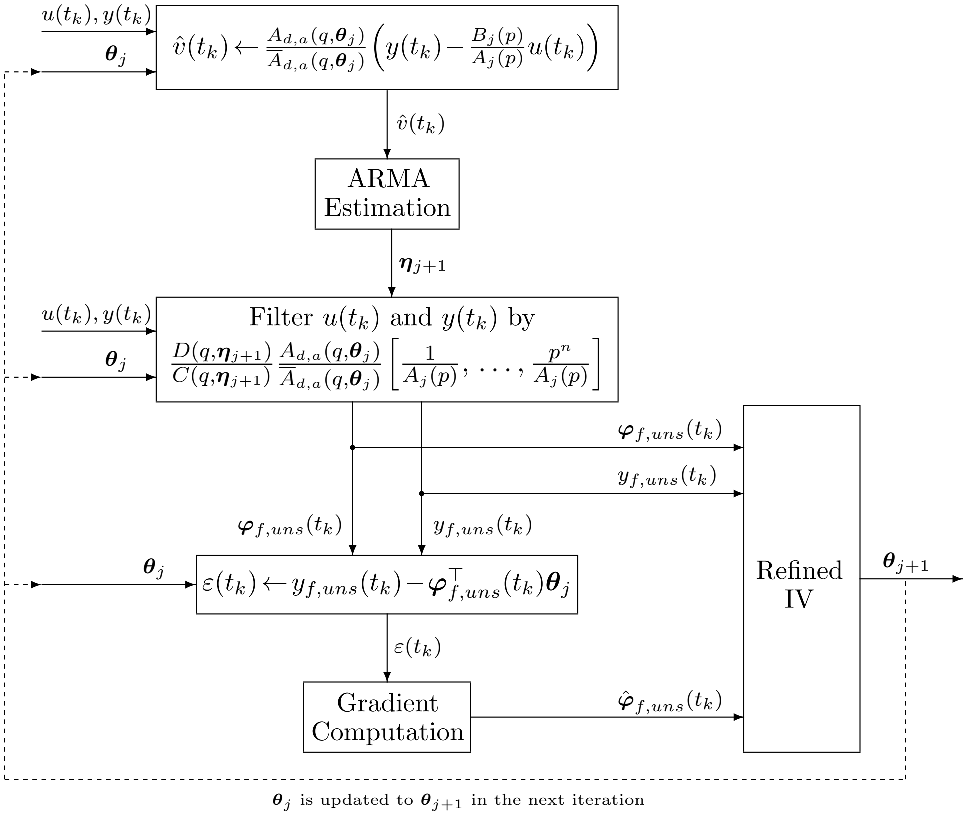

Figure 1. Implementation of the extended RIVC algorithm for when the input has a ZOH or FOH behavior. Note that in this case, the filtering process is done in discrete-time by discretizing the bank of continuous-time filters $p^i/A_j(p)$ and cascading them with the all-pass filter and inverse noise model. The algorithm is initialized with $\boldsymbol{\theta}_j = \boldsymbol{\theta}_1$ (i.e., $j = 1$). Note that the extended SRIVC algorithm follows from omitting the ARMA estimation step and using $D(q) = C(q) = 1$.

The computation of (26) is not direct due to the mixture of continuous and discrete-time transfer functions. For simplicity, we will only analyze the case when the input has a ZOH or FOH intersample behavior. For the numerator components present in $\boldsymbol{\theta}$, the derivative can be directly evaluated as

$$\frac{\partial}{\partial b_l}\hat{y}(t_k|\boldsymbol{\theta}, \boldsymbol{\eta}) = \frac{\tilde{D}(q, \boldsymbol{\eta}, \boldsymbol{\theta})}{\tilde{C}(q, \boldsymbol{\eta}, \boldsymbol{\theta})}\frac{p^l}{A(p, \boldsymbol{\theta})}u(t_k), \quad l = 0, 1, \dots, m.$$

In other words, the last $m+1$ components of $\hat{\boldsymbol{\varphi}}_f(t_k, \boldsymbol{\theta})$ coincide with those of the regressor $\boldsymbol{\varphi}_f(t_k, \boldsymbol{\theta})$. The first n components of $\hat{\boldsymbol{\varphi}}_f(t_k, \boldsymbol{\theta})$, associated with the denominator polynomial of the model, are more difficult to compute. In general, they are

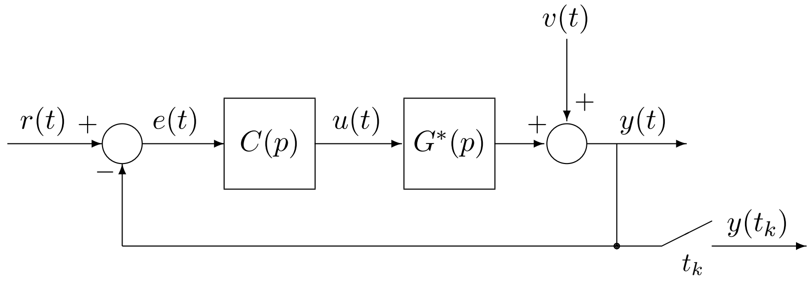

Figure 2. Block diagram for the closed-loop framework.

given by

$$\frac{\partial}{\partial a_l}\hat{y}(t_k|\boldsymbol{\theta},\boldsymbol{\eta}) = \frac{D(q,\boldsymbol{\eta})}{C(q,\boldsymbol{\eta})}\frac{\partial}{\partial a_l}\left\{\frac{A_{d,a}(q,\boldsymbol{\theta})}{\overline{A}_{d,a}(q,\boldsymbol{\theta})}\left(\left\{\frac{B(p,\boldsymbol{\theta})}{A(p,\boldsymbol{\theta})}u(t)\right\}_{t=t_k} - y(t_k)\right)\right\}, \quad l = 1, 2, \ldots, n. \tag{27}$$

For most practical applications, computing the derivative via numerical approximation is sufficient. If a gradient is needed in explicit form, the computations are provided in Appendix D.

5. The extended CLSRIVC method

In this section, we extend the ideas of Section 4 to study indirect closed-loop continuous-time system identification when the open-loop plant is unstable. We will not estimate the noise model in this case, although our approach can also be extended to handle that scenario by modifying the CLRIVC method in Chapter 5 of Garnier and Wang (2008). Let us now consider the closed-loop scheme in Fig. 2, where

$$\begin{aligned} y(t) &= \frac{G^*(p)C(p)}{1+G^*(p)C(p)}r(t) + \frac{1}{1+G^*(p)C(p)}v(t), \\ u(t) &= \frac{C(p)}{1+G^*(p)C(p)}r(t) - \frac{C(p)}{1+G^*(p)C(p)}v(t). \end{aligned}$$

In this framework, $v(t)$ is a stochastic process that is assumed to be constant between sampling instants[3]. The model estimates of each iteration of the CLSRIVC estimator using sampled data are given by (9), but where the instrument vector is now constructed with filtered versions of the reference signal:

$$\hat{\boldsymbol{\varphi}}_f(t_k) = \left[\frac{-pT_{o,j}(p)}{A_j(p)}r(t), \ldots, \frac{-p^n T_{o,j}(p)}{A_j(p)}r(t), \frac{S_{uo,j}(p)}{A_j(p)}r(t), \ldots, \frac{p^m S_{uo,j}(p)}{A_j(p)}r(t)\right]_{t=t_k}^\top, \tag{28}$$

with the sensitivity functions $T_{o,j}(p)$ and $S_{uo,j}(p)$ being defined by

$$T_{o,j}(p) := \frac{G_j(p)C(p)}{1+G_j(p)C(p)}; \quad S_{uo,j}(p) := \frac{C(p)}{1+G_j(p)C(p)}.$$

[3]This formulation, which results in a mixture between continuous-time and discrete-time signals, is commonly used in the closed-loop continuous-time system identification literature; see e.g. Chapter 5 of Garnier and Wang (2008).

Note that, following Remark 1, in (28) we consider a generalized version of the CLSRIVC estimator where the reference is allowed to be any arbitrary continuous-time signal.

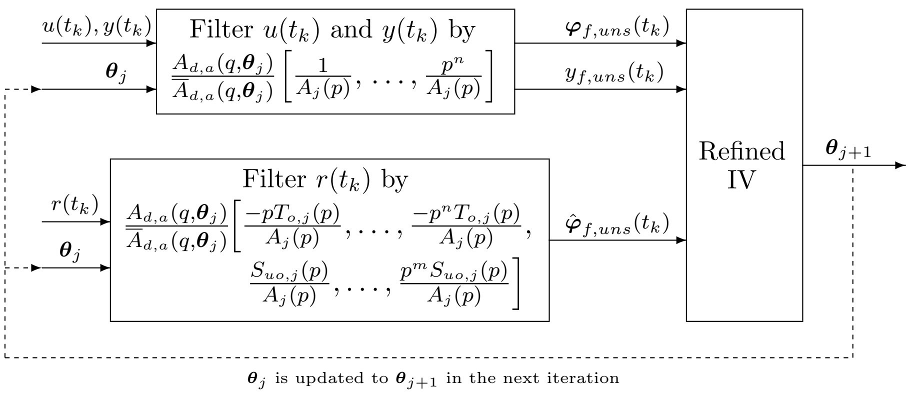

Figure 3. Implementation of the extended CLSRIVC algorithm for when the input has a ZOH or FOH behavior. The algorithm is initialized with $\boldsymbol{\theta}_j = \boldsymbol{\theta}_1$ (i.e., $j = 1$).

The proposed solution to the instability problem in CLSRIVC is illustrated in Figure 3. The idea is to use $y_{f,uns}(t_k, \boldsymbol{\theta}_j)$ and $\boldsymbol{\varphi}_{f,uns}(t_k, \boldsymbol{\theta}_j)$ as in (21) and (22), respectively (but with $C_j(q) = D_j(q) = 1$), and to construct the modified instrument vector

$$\hat{\boldsymbol{\varphi}}_{f,uns}(t_k) = \frac{A_{d,a}(q, \boldsymbol{\theta}_j)}{\bar{A}_{d,a}(q, \boldsymbol{\theta}_j)} \hat{\boldsymbol{\varphi}}_f(t_k), \tag{29}$$

where $\hat{\boldsymbol{\varphi}}_f(t_k)$ is defined in (28). Again, the iterations are given by (25). If the input or reference are arbitrary signals, the filtering procedure follows the same ideas discussed in Appendix B. For the case of ZOH or FOH reference and input signals, which implies that there exists a hold device between the controller and the plant[4], Lemma 5.1 characterizes the asymptotic behavior of the estimator.

Lemma 5.1. *Consider the extended CLSRIVC method with filtered output* (21)*, filtered regressor vector* (22) *and filtered instrument vector* (29)*, and assume that the reference, input, and noise are ZOH or FOH signals and are stationary. Furthermore, assume that the extended CLSRIVC method converges for all N large enough, and that the matrix*

$$\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_{f,uns}(t_k)\boldsymbol{\varphi}_{f,uns}^\top(t_k)\right\} \tag{30}$$

is non-singular. Then, at the converging point and as $N \to \infty$, the extended CLSRIVC method and its standard version (i.e., (9) *with* (28) *as filtered instrument vector) solve the same pseudo-linear regression equation, namely,*

$$\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k, \bar{\boldsymbol{\theta}})\left(y(t_k) - \frac{\bar{B}(p)}{\bar{A}(p)}u(t_k)\right)\right\} = \mathbf{0}.$$

Proof. See Appendix E. □

[4]Note that in this case, the sensitivity functions can be described in discrete-time, and the filtered instrument is formed by cascading the sensitivity functions with the discrete-time equivalent of the bank of filters $\left[\frac{1}{A_j(p)}, \dots, \frac{p^n}{A_j(p)}\right]$.

Remark 4. From the consistency analysis in González et al. (2021), we know that the standard CLSRIVC method is generically consistent (with respect to the model and system parameters) even when the noise is not white, as long as it is uncorrelated with the reference signal. Thus, under that condition and the assumptions in Lemma 5.1, we find that the extended CLSRIVC method is generically consistent for both stable and unstable systems. Note that, provided some identifiability conditions are met, the matrix in (30) can be shown to be generically non-singular following the same approach as in Pan, Welsh, Gonzalez, and Rojas (2021).

Remark 5. The three-stage method for closed-loop continuous-time system identification presented by Young et al. (2009) is claimed to induce statistical efficiency for stable open-loop systems. However, under its current implementation it cannot be used when the open-loop system is unstable. The all-pass filtering technique exploited in the present article can also be applied to this method, thus extending it to open-loop unstable systems as well. After the generation of an estimate of the noise-free input $\hat{u}(t_k)$, the extended RIVC or SRIVC methods presented in Section 4 permit the identification of an unstable estimate $\hat{G}(p) = \hat{B}(p)/\hat{A}(p)$ using $\hat{u}(t_k)$ and $y(t_k)$. Finally, a refinement of this model estimate can be obtained based on the filtered estimates of the noise-free input and output:

$$\tilde{u}(t_k) = \frac{A_{d,a}(q, \hat{\boldsymbol{\theta}})}{\overline{A}_{d,a}(q, \hat{\boldsymbol{\theta}})}\hat{u}(t_k), \text{ and } \tilde{y}(t_k) = \frac{A_{d,a}(q, \hat{\boldsymbol{\theta}})}{\overline{A}_{d,a}(q, \hat{\boldsymbol{\theta}})}\Big(y(t_k) - \hat{G}(p)\big[u(t_k) - \hat{u}(t_k)\big]\Big).$$

6. Estimating continuous-time systems with integral action

In many applications, such as platoon vehicle modeling (Ploeg, Scheepers, Van Nunen, Van de Wouw, & Nijmeijer, 2011), modeling of DC motors (Ljung, Glad, & Hansson, 2021), or position tracking (Goodwin, Graebe, & Salgado, 2001), prior knowledge of the system's dynamics suggests that there is an integrator in the plant. Unfortunately, the all-pass filtering procedure introduced in Section 4 only stabilizes strictly unstable poles of the model. Here we propose two solutions for identifying systems with integration.

Solution 1 (Use the extended versions of LSSVF, SRIVC, RIVC, and CLSRIVC for a monic denominator polynomial of the model): Instead of deriving the extended estimators using an anti-monic denominator polynomial for the system (as in (1)), we can consider a monic polynomial for $A^*(p)$ of the form

$$A^*(p) = p^{n^*} + a^*_{n^*-1}p^{n^*-1} + \cdots + a^*_1 p + a^*_0.$$

With this description, the regression equation is still of the form (20), but with

$$y_{f,uns}(t_k) = \frac{\tilde{D}(q, \boldsymbol{\eta}, \boldsymbol{\theta})}{\tilde{C}(q, \boldsymbol{\eta}, \boldsymbol{\theta})}\frac{p^n}{A(p, \boldsymbol{\theta})}y(t_k),$$

$$\boldsymbol{\varphi}^{\top}_{f,uns}(t_k) = \frac{\tilde{D}(q,\boldsymbol{\eta},\boldsymbol{\theta})}{\tilde{C}(q,\boldsymbol{\eta},\boldsymbol{\theta})}\left[\frac{-1}{A(p,\boldsymbol{\theta})}y(t_k), \ldots, \frac{-p^{n-1}}{A(p,\boldsymbol{\theta})}y(t_k), \left\{\frac{1}{A(p,\boldsymbol{\theta})}u(t)\right\}_{t=t_k}, \ldots, \left\{\frac{p^m}{A(p,\boldsymbol{\theta})}u(t)\right\}_{t=t_k}\right].$$

The extended SRIVC and RIVC estimators with monic denominator polynomial for the model are obtained by computing (25) iteratively, and the extended CLSRIVC estimator for this case uses the filtered instrument vector

$$\hat{\boldsymbol{\varphi}}_f(t_k) = \left[\frac{-T_{o,j}(p)}{A_j(p)}r(t), \dots, \frac{-p^{n-1}T_{o,j}(p)}{A_j(p)}r(t), \frac{S_{uo,j}(p)}{A_j(p)}r(t), \dots, \frac{p^m S_{uo,j}(p)}{A_j(p)}r(t)\right]^\top_{t=t_k}.$$

The advantage of this formulation is that the estimate for a_0^* can go to zero, contrary to the anti-monic formulation which fixes the constant term of the denominator to one. The drawback of these methods is that integration is not forced explicitly, which can lead to an increase in the covariance of the parameter estimate compared to estimators that can fix an integrator in the model, as the ones proposed in Solution 2. Note that the anti-monic formulation studied in Sections 3, 4 and 5 is favored when the model structure is not known, since it can lead to consistent estimators despite possible over-parametrization in the denominator (Pan, González, et al., 2020).

Solution 2 (Force a pole at zero in the extended versions of SRIVC, RIVC and CLSRIVC): The key idea behind this approach is that, if the closed-loop system is stable and the system has an integrator, the mean value of the system input is zero. In other words, filtering the input $u(t_k)$ through a filter with integration yields a bounded signal, which can be used in the construction of the filtered regressor and instrument vectors. The procedure proposed here resembles the one in Pan, Welsh, Brichta, Drury, and Stoddard (2020), where a known stable pole was forced into the SRIVC algorithm to identify a biological system.

For simplicity, assume that the system has one integrator. If the true system is of the form $B^*(p)/[p\tilde{A}^*(p)]$, where $\tilde{A}^*(p)$ is a monic polynomial of order n, then the extended RIVC algorithm with a fixed pole at $p = 0$ can be obtained from the regression equation (20), where

$$y_{f,uns}(t_k) = \frac{\tilde{D}(q,\boldsymbol{\eta},\boldsymbol{\theta})}{\tilde{C}(q,\boldsymbol{\eta},\boldsymbol{\theta})}\frac{p^{n-1}}{\tilde{A}(p,\boldsymbol{\theta})}y(t_k),$$

$$\boldsymbol{\varphi}^\top_{f,uns}(t_k) = \frac{\tilde{D}(q,\boldsymbol{\eta},\boldsymbol{\theta})}{\tilde{C}(q,\boldsymbol{\eta},\boldsymbol{\theta})}\left[\frac{-1}{\tilde{A}(p,\boldsymbol{\theta})}y(t_k), \dots, \frac{-p^{n-2}}{\tilde{A}(p,\boldsymbol{\theta})}y(t_k), \left\{\frac{1}{p\tilde{A}(p,\boldsymbol{\theta})}u(t)\right\}_{t=t_k}, \dots, \left\{\frac{p^{m-1}}{\tilde{A}(p,\boldsymbol{\theta})}u(t)\right\}_{t=t_k}\right],$$

and the extended CLSRIVC estimator for this case has an instrument vector given by

$$\hat{\boldsymbol{\varphi}}_f(t_k) = \left[\frac{-T_{o,j}(p)}{\tilde{A}_j(p)}r(t), \dots, \frac{-p^{n-2}T_{o,j}(p)}{\tilde{A}_j(p)}r(t), \frac{S_{uo,j}(p)}{p\tilde{A}_j(p)}r(t), \dots, \frac{p^{m-1}S_{uo,j}(p)}{\tilde{A}_j(p)}r(t)\right]^\top_{t=t_k}.$$

These formulations lead to model estimates with a fixed pole at $p = 0$, and also admit other unstable poles in the estimation process, if present and known a priori.

7. Simulations

In this section we show the applicability of the proposed methods to identify unstable continuous-time systems. For the first three tests, we consider the following unstable plant:

$$G^*(p) = \frac{-0.5p + 1}{p^2 + 2\xi\omega_n p + \omega_n^2}, \tag{31}$$

where $\omega_n^2 = 2$ and $\xi = -0.05$. The output of this system is sampled at $h = 0.1$[s], unless indicated otherwise.

7.1. Discrete-time control, output error model structure

First, the system in (31) is controlled by the discrete-time PID feedback controller

$$C(q) = 0.0272 + \frac{2.54 \cdot 10^{-5}}{1 - q^{-1}} + 7.29(1 - q^{-1}),$$

which includes a digital-to-analog device at its output, rendering a continuous-time input $u(t)$ that is constant between samples. The noise is also assumed to be constant between samples, and thus the input and output samples can be generated entirely in discrete-time by

$$\begin{aligned} y(t_k) &= \frac{G_d^*(q)C(q)}{1 + G_d^*(q)C(q)} r(t_k) + \frac{1}{1 + G_d^*(q)C(q)} v(t_k), \\ u(t_k) &= \frac{C(q)}{1 + G_d^*(q)C(q)} r(t_k) - \frac{C(q)}{1 + G_d^*(q)C(q)} v(t_k), \end{aligned}$$

where $G_d^*(q)$ is the ZOH equivalent of $G^*(p)$, and $r(t_k)$ with $v(t_k)$ are white noise signals of variance 1 and 0.05, respectively.

Data from 300 Monte Carlo runs, each with $N = 15000$ input-output samples, are used for identification. Four algorithms that have been introduced in this work are evaluated: LSSVF for unstable systems (labeled LSSVF-uns), a refined least-squares method (labeled Refined LS-uns), SRIVC and closed-loop SRIVC for unstable systems (SRIVC-uns and CLSRIVC-uns, respectively). These estimators are compared against the standard SRIVC estimator as implemented in Algorithm 1 of González et al. (2021). The LSSVF-uns estimate is given by one iteration of (24), and is initialized with a filter of the form $1/(\tilde{a}_2 p^2 + \tilde{a}_1 p + 1)$, where $\tilde{a}_1$ and $\tilde{a}_2$ are the true parameters perturbed by noise of standard deviation 0.02 and 0.1, respectively. The refined LS-uns, SRIVC-uns and CLSRIVC-uns estimators are computed based on the initialization given by LSSVF-uns, and the instrument vector of the SRIVC-uns algorithm is computed by a first-order numerical differentiation with step size 10^{-8} (Stoer & Bulirsch, 2002). The standard SRIVC estimator is initialized with the LSSVF-uns estimate but with its unstable poles reflected into the stable region of the complex plane. The maximum number of iterations of all algorithms is set to 50, with a tolerance factor of 10^{-8}, i.e., they terminate when

$$\frac{\|\boldsymbol{\theta}_{j+1} - \boldsymbol{\theta}_j\|_2}{\|\boldsymbol{\theta}_j\|_2} < 10^{-8}.$$

The mean and standard deviation of the estimated parameters are given in Table 1, and 10 realizations of both the SRIVC and SRIVC-uns estimators are provided in Figure 4. Out of the 300 runs, the SRIVC-uns failed to produce an estimate in one realization (which was not considered in the statistics), while the other methods did not have errors. The LSSVF-uns method provides a reasonable estimate for the initialization of the SRIVC-uns and CLSRIVC-uns methods, which achieve great accuracy in the estimated parameters. These results mimic the behavior of their stable counter-

Method	Statistic	a_1 (-0.0707)	a_2 (0.5)	b_0 (0.5)	b_1 (-0.25)
SRIVC	Mean	0.1132	0.5071	0.5059	-0.2555
	S. Dev.	$2.36 \cdot 10^{-1}$	$4.50 \cdot 10^{-2}$	$3.46 \cdot 10^{-1}$	$4.45 \cdot 10^{-2}$
LSSVF-uns	Mean	-0.0614	0.4672	0.4523	-0.2252
	S. Dev.	$8.95 \cdot 10^{-3}$	$1.24 \cdot 10^{-2}$	$5.27 \cdot 10^{-2}$	$9.65 \cdot 10^{-3}$
Refined LS-uns	Mean	-0.0669	0.4726	0.4569	-0.2280
	S. Dev.	$1.74 \cdot 10^{-3}$	$2.70 \cdot 10^{-3}$	$6.29 \cdot 10^{-3}$	$2.03 \cdot 10^{-3}$
SRIVC-uns	Mean	-0.0708	0.5001	0.5002	-0.2501
	S. Dev.	$1.49 \cdot 10^{-3}$	$1.00 \cdot 10^{-3}$	$3.82 \cdot 10^{-3}$	$1.17 \cdot 10^{-3}$
CLSRIVC-uns	Mean	-0.0708	0.5001	0.5002	-0.2501
	S. Dev.	$1.53 \cdot 10^{-3}$	$1.12 \cdot 10^{-3}$	$3.96 \cdot 10^{-3}$	$2.10 \cdot 10^{-3}$

Table 1. Mean and standard deviation of the estimated model parameters obtained with 300 Monte Carlo runs with a discrete-time controller and an output error model structure.

parts for stable system identification, as it can be proven that, provided some technical conditions are met[5], CLSRIVC is generically consistent for white or colored noise while SRIVC is generically consistent in closed-loop if the disturbance is white. On the other hand, the refined least-squares estimate is more precise than its one-iteration counterpart but does not minimize the 2-norm of the prediction error, which leads to biasedness in this example. As seen in Figure 4, the standard SRIVC shows poor performance in the phase plots compared to the SRIVC-uns estimates, since it only computes stable estimates. Furthermore, the magnitude plot for the SRIVC estimates show that the realizations are much more volatile than the SRIVC-uns counterparts, which is in agreement with the large standard deviations of the SRIVC parameters displayed in Table 1.

7.2. Continuous-time control, output error model structure

We now test the performance of the proposed SRIVC estimator for arbitrary input signal excitation. This time, a continuous-time PID controller is used to control the system in (31) of the form

$$C(p) = 0.0249 + \frac{1.42 \cdot 10^{-4}}{p} + \frac{0.974p}{0.00897p + 1}.$$

Two sampling periods are tested, $h = 0.1$[s] and $h = 0.02$[s]. The total time of both experiments is the same ($500[s]$), and the statistics of the reference signal and disturbance noise are the same for both experiments. The goal is to test the effect of the sampling period on the biasedness of the extended LSSVF and SRIVC estimators. For this, a test of 300 Monte Carlo runs are performed for each scenario.

Table 2 presents the mean and standard deviation of the parameters for the LSSVF-uns and SRIVC-uns estimators adapted for arbitrary input signals (labeled LSSVF-os-uns and SRIVC-os-uns, respectively). This adaptation is done following the Delta-domain mechanism proposed in Appendix B. The results show that the SRIVC-uns estimator is preferred over its initialization method LSSVF-uns, and that the bias of SRIVC-uns is reduced when the sampling period decreases. This is in line with

[5]The complete result mentioned here exceeds the scope of this work, and will soon be presented as another contribution.

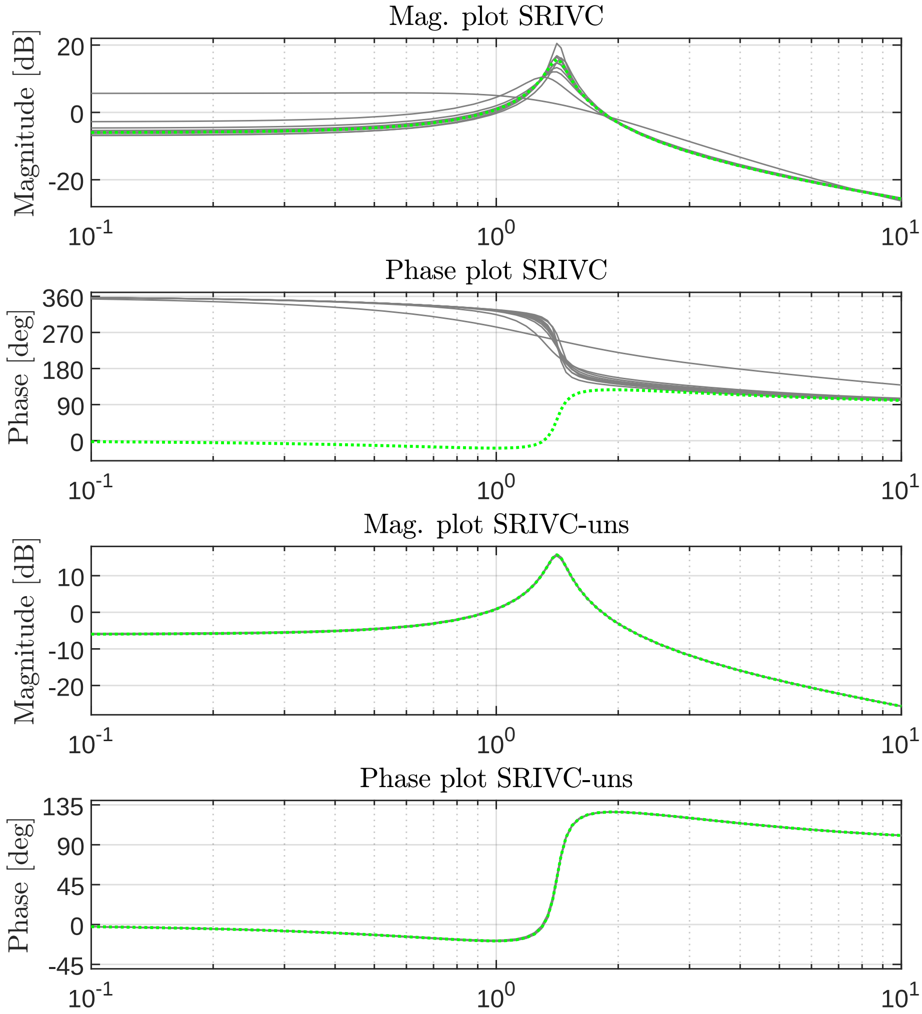

Figure 4. Bode plots for 10 realizations of the SRIVC estimator (grey, upper two plots), 10 realizations of the SRIVC-uns estimator (grey, lower two plots), and the true system (dashed, green).

h	Method	Stats.	$a_1(-0.0707)$	a_2 (0.5)	b_0 (0.5)	b_1 (-0.25)
0.1[s]	LSSVF-os-uns	Mean	-0.0642	0.4944	0.4912	-0.2512
		S. Dev.	$4.01 \cdot 10^{-3}$	$4.62 \cdot 10^{-3}$	$2.06 \cdot 10^{-2}$	$2.72 \cdot 10^{-3}$
	SRIVC-os-uns	Mean	-0.0661	0.5018	0.5018	-0.2553
		S. Dev.	$1.20 \cdot 10^{-3}$	$7.23 \cdot 10^{-4}$	$2.45 \cdot 10^{-3}$	$5.58 \cdot 10^{-4}$
0.02[s]	LSSVF-os-uns	Mean	-0.0676	0.4929	0.4894	-0.2485
		S. Dev.	$3.75 \cdot 10^{-3}$	$3.88 \cdot 10^{-3}$	$1.28 \cdot 10^{-2}$	$2.10 \cdot 10^{-3}$
	SRIVC-os-uns	Mean	-0.0701	0.5003	0.4991	-0.2527
		S. Dev.	$1.24 \cdot 10^{-3}$	$8.40 \cdot 10^{-4}$	$2.55 \cdot 10^{-3}$	$5.56 \cdot 10^{-4}$

Table 2. Mean and standard deviation of the estimated model parameters obtained with 300 Monte Carlo runs with a continuous-time controller and an output error model structure.

Method	Stats.	$a_1(-0.0707)$	$a_2(0.5)$	$b_0(0.5)$	$b_1(-0.25)$	$c_1(0.5)$	$d_1(-0.85)$
RIVC	Mean	0.1593	0.4910	0.6045	-0.2532	0.5435	-0.9252
	S. Dev.	$1.96\cdot10^{-2}$	$6.20\cdot10^{-3}$	$2.54\cdot10^{-2}$	$3.67\cdot10^{-3}$	$3.24\cdot10^{-3}$	$8.98\cdot10^{-3}$
LSSVF-uns	Mean	-0.0645	0.4806	0.3976	-0.2436		
	S. Dev.	$1.20\cdot10^{-2}$	$1.15\cdot10^{-2}$	$3.96\cdot10^{-2}$	$7.83\cdot10^{-3}$		
SRIVC-uns	Mean	-0.1022	0.5026	0.4807	-0.2616		
	S. Dev.	$7.66\cdot10^{-3}$	$2.96\cdot10^{-3}$	$1.29\cdot10^{-2}$	$2.50\cdot10^{-3}$		
CLSRIVC-uns	Mean	-0.0707	0.5003	0.5004	-0.2499		
	S. Dev.	$5.52\cdot10^{-3}$	$3.72\cdot10^{-3}$	$1.21\cdot10^{-2}$	$7.04\cdot10^{-3}$		
RIVC-uns	Mean	-0.0701	0.5003	0.4998	-0.2501	0.4998	-0.8494
	S. Dev.	$4.35\cdot10^{-3}$	$2.74\cdot10^{-3}$	$8.43\cdot10^{-3}$	$1.43\cdot10^{-3}$	$4.77\cdot10^{-3}$	$7.38\cdot10^{-3}$

Table 3. Mean and standard deviation of the estimated system and noise model parameters obtained with 300 Monte Carlo runs with a discrete-time controller and a hybrid Box-Jenkins model structure.

the analysis in Appendix B, which indicates that the misspecification of the model structure (i.e., (B1) instead of (15)) is less severe with small sampling periods. Note that a fast sampling rate for accurate estimates is only required for the continuous-time control case, as the model structure we use in this scenario cannot enforce stable predictors and also be equivalent to (4) at the same time.

7.3. Discrete-time control, hybrid Box-Jenkins model structure

Instead of white noise, we now consider a colored noise in the loop and include it in the identification procedure. The discrete-time ARMA noise process is of the form

$$v(t_k) = \frac{1+0.5q^{-1}}{1-0.85q^{-1}}e(t_k),$$

where $e(t_k)$ is white noise of variance 0.01. Just as in the previous simulation examples, 15000 samples are obtained for identification and 300 Monte Carlo runs are performed. The LSSVF-uns estimator is used as an initialization point for the RIVC, SRIVC-uns, CLSRIVC-uns and RIVC-uns estimators, and the ARMA estimation step of the RIVC-uns estimator is computed using the `armax` command of the System Identification Toolbox in MATLAB (Ljung & Singh, 2012).

In Table 3 we present the mean and standard deviation of each parameter for each estimator under study. The SRIVC-uns estimator fails to provide accurate estimates due to the colored noise, which introduces bias in the method. This is not a problem for the CLSRIVC-uns estimator, as it is generically consistent even if the disturbance is colored. Again, these results mimic the behavior of the standard version of these estimators for stable, continuous-time system identification. The RIVC-uns method achieves excellent performance for estimating unstable systems in a hybrid BJ model structure, while the standard RIVC method is biased and does not deliver unstable models.

7.4. System with integral action

In this final test, we identify a continuous-time system with integral action that is studied in both open and closed-loop. The system and controller we consider in this

Method	Stats.	a_0	a_1	b_0	b_1
LSSVF-uns-1	Bias MSE	$4.24 \cdot 10^{-7}$ $7.63 \cdot 10^{-11}$	$4.86 \cdot 10^{-3}$ $4.36 \cdot 10^{-4}$	$-9.24 \cdot 10^{-3}$ $1.18 \cdot 10^{-3}$	$6.68 \cdot 10^{-3}$ $7.97 \cdot 10^{-4}$
SRIVC-uns-1	Bias MSE	$-2.88 \cdot 10^{-8}$ $2.31 \cdot 10^{-12}$	$4.43 \cdot 10^{-4}$ $1.08 \cdot 10^{-4}$	$-9.09 \cdot 10^{-4}$ $4.30 \cdot 10^{-4}$	$1.22 \cdot 10^{-3}$ $5.08 \cdot 10^{-4}$
SRIVC-uns-2	Bias MSE	0 0	$1.65 \cdot 10^{-4}$ $1.10 \cdot 10^{-4}$	$-3.82 \cdot 10^{-4}$ $4.37 \cdot 10^{-4}$	$-8.25 \cdot 10^{-4}$ $5.04 \cdot 10^{-4}$
CLSRIVC-uns-1	Bias MSE	$6.60 \cdot 10^{-5}$ $7.28 \cdot 10^{-7}$	$-3.73 \cdot 10^{-3}$ $1.50 \cdot 10^{-3}$	$3.53 \cdot 10^{-3}$ $3.27 \cdot 10^{-3}$	$-3.85 \cdot 10^{-3}$ $1.35 \cdot 10^{-3}$
CLSRIVC-uns-2	Bias MSE	0 0	$-3.42 \cdot 10^{-3}$ $1.47 \cdot 10^{-3}$	$3.16 \cdot 10^{-3}$ $3.24 \cdot 10^{-3}$	$-3.62 \cdot 10^{-3}$ $1.33 \cdot 10^{-3}$

Table 4. Bias and MSE of the estimated model parameters obtained from a system with integral action with 300 Monte Carlo runs.

test are given by

$$G^*(p) = \frac{p-2}{p(p+1)}, \quad C(q) = -0.249 - 2.07(1-q^{-1}),$$

with sampling period $h = 0.1[s]$. The additive noise $v(t_k)$ is a zero-mean Gaussian white noise of variance 0.2. Two experimental conditions are tested: open-loop (i.e., without controller, and with input $u(t_k)$ being a zero-mean Gaussian noise of variance 1), and closed-loop (with the reference $r(t_k)$ also being zero-mean Gaussian white noise with variance 1). The same number of samples and Monte Carlo runs as in the previous test are used to obtain estimates of the extended versions of LSSVF and SRIVC for open-loop, and CLSRIVC for closed-loop. The estimators that are computed following Solution 1 of Section 6 are labeled as LSSVF-uns-1, SRIVC-uns-1 and CLSRIVC-uns-1, and the ones computed from Solution 2 are labeled SRIVC-uns-2 and CLSRIVC-uns-2. The LSSVF-uns-1 estimator is initialized with a random polynomial $A_1(p) = p^2 + \tilde{a}_1 p + \tilde{a}_0$, with $\tilde{a}_0$ and $\tilde{a}_1$ being the true parameters perturbed by white noise of standard deviation 0.1. The model from this estimator is used as initialization for the extended SRIVC and CLSRIVC algorithms.

The bias and MSE (mean square error) of each parameter is recorded and shown in Table 4. Forcing one pole to zero leads to a decrease in bias for the open and closed loop tests, while having similar MSE compared to the SRIVC and CLSRIVC variants obtained from Solution 1. At any rate, we verify that both solutions introduced in Section 6 are adequate for estimating systems with integration in open and closed-loop. The SRIVC and CLSRIVC estimators using Solution 1 are also seen to be robust despite the perturbation noise being of high variance, thanks to the ad-hoc stable filtering that is performed whenever the previous iteration returns an unstable estimate.

8. Conclusions

In this work, we have presented extensions of refined instrumental variable methods that allow the direct and indirect identification of unstable continuous-time systems in closed-loop. The main technique consists of considering an ad-hoc noise model which iteratively cancels the unstable poles present in the model iterates. This tool has been applied successfully to the SRIVC and RIVC estimators, as well as to its initialization method, the LSSVF estimator. The same technique has been shown to be promising for indirect closed-loop identification with the CLSRIVC estimator. These estimators have also been extended to identify systems with integral action. In the case of continuous-time controllers without a hold device at their output, the approach we propose allows unstable models at the expense of bias in the parameters, and one way to mitigate this bias is to reduce the sampling period.

Appendix A. Proof of Proposition 4.1

Proof. The model (15) can also be written as

$$y(t_k) = \left\{ \frac{B(p,\boldsymbol{\theta})}{A(p,\boldsymbol{\theta})} u(t) \right\}_{t=t_k} + \left(\frac{\tilde{C}(q,\boldsymbol{\eta},\boldsymbol{\theta})}{\tilde{D}(q,\boldsymbol{\eta},\boldsymbol{\theta})} - 1 \right) \varepsilon(t_k) + \varepsilon(t_k).$$

On the other hand,

$$\varepsilon(t_k) = \frac{\tilde{D}(q,\boldsymbol{\eta},\boldsymbol{\theta})}{\tilde{C}(q,\boldsymbol{\eta},\boldsymbol{\theta})} \left(y(t_k) - \left\{ \frac{B(p,\boldsymbol{\theta})}{A(p,\boldsymbol{\theta})} u(t) \right\}_{t=t_k} \right),$$

which leads to

$$\begin{aligned} y(t_k) &= \left\{ \frac{B(p,\boldsymbol{\theta})}{A(p,\boldsymbol{\theta})} u(t) \right\}_{t=t_k} + \left(\frac{\tilde{C}(q,\boldsymbol{\eta},\boldsymbol{\theta})}{\tilde{D}(q,\boldsymbol{\eta},\boldsymbol{\theta})} - 1 \right) \frac{\tilde{D}(q,\boldsymbol{\eta},\boldsymbol{\theta})}{\tilde{C}(q,\boldsymbol{\eta},\boldsymbol{\theta})} \left(y(t_k) - \left\{ \frac{B(p,\boldsymbol{\theta})}{A(p,\boldsymbol{\theta})} u(t) \right\}_{t=t_k} \right) + \varepsilon(t_k) \\ &= \frac{\tilde{D}(q,\boldsymbol{\eta},\boldsymbol{\theta})}{\tilde{C}(q,\boldsymbol{\eta},\boldsymbol{\theta})} \left\{ \frac{B(p,\boldsymbol{\theta})}{A(p,\boldsymbol{\theta})} u(t) \right\}_{t=t_k} + \left(1 - \frac{\tilde{D}(q,\boldsymbol{\eta},\boldsymbol{\theta})}{\tilde{C}(q,\boldsymbol{\eta},\boldsymbol{\theta})} \right) y(t_k) + \varepsilon(t_k). \end{aligned} \tag{A1}$$

The one-step-ahead optimal predictor is given by the conditional expectation of $y(t_k)$ given the previous values of the input and output. Since $\varepsilon(t_k)$ is white noise, and the right-hand side of (A1) depends solely on previous values of $y(t_k)$ and $u(t)$, the result follows. □

Appendix B. Filtered regressor and output when the input is an arbitrary signal

The following filtered regressor and output expressions are proposed to solve the instability problem when the input is an arbitrary signal that can be over-sampled. Instead of considering the all-pass filter $A_{d,a}(q,\boldsymbol{\theta})/\overline{A}_{d,a}(q,\boldsymbol{\theta})$ for prefiltering the input, we introduce its fast-sampled Delta version $A_{\Delta,a}(\delta,\boldsymbol{\theta})/\overline{A}_{\Delta,a}(\delta,\boldsymbol{\theta})$, where if $1/A(p)$ has unstable poles $\{p_i\}_{i=1}^{n_a}$, the polynomial $A_{\Delta,a}(\delta,\boldsymbol{\theta})$ has roots of the form

$$\delta_i = \frac{S}{h} \left(e^{\frac{p_i h}{S}} - 1 \right), \quad i = 1, \dots, n_a.$$

Just like its q-domain version, the polynomial $\overline{A}_{\Delta,a}(\delta,\boldsymbol{\theta})$ is designed such that the filter $A_{\Delta,a}(\delta,\boldsymbol{\theta})/\overline{A}_{\Delta,a}(\delta,\boldsymbol{\theta})$ is all-pass. This means that $\overline{A}_{\Delta,a}(\delta,\boldsymbol{\theta})$ must be given by

$$\overline{A}_{\Delta,a}(\delta,\boldsymbol{\theta}) = \prod_{i=1}^{n_a}\left(\delta_i + \delta\left[1+\frac{h}{S}\delta_i\right]\right).$$

We denote the over-sampled input by $\tilde{u}$. With this in mind, we can fit an ARMA model to the estimated noise sequence

$$\hat{v}(t_k) = \frac{A_{d,a}(q,\boldsymbol{\theta}_j)}{\overline{A}_{d,a}(q,\boldsymbol{\theta}_j)}y(t_k) - \frac{A_{\Delta,a}(\delta,\boldsymbol{\theta}_j)}{\overline{A}_{\Delta,a}(\delta,\boldsymbol{\theta}_j)}\frac{B_\Delta(\delta,\boldsymbol{\theta}_j)}{A_\Delta(\delta,\boldsymbol{\theta}_j)}\tilde{u}(t_k),$$

where $B_\Delta(\delta,\boldsymbol{\theta}_j)/A_\Delta(\delta,\boldsymbol{\theta}_j)$ is the Delta-domain equivalent of the continuous-time transfer function $B(p,\boldsymbol{\theta}_j)/A(p,\boldsymbol{\theta}_j)$ (Middleton & Goodwin, 1990). After this estimation step, we compute all the filtered signals in the Delta domain as

$$\begin{aligned} y_{f,uns}(t_k) &= \frac{D(q,\boldsymbol{\eta}_{j+1})}{C(q,\boldsymbol{\eta}_{j+1})}\frac{A_{d,a}(q,\boldsymbol{\theta}_j)}{\overline{A}_{d,a}(q,\boldsymbol{\theta}_j)}\frac{1}{A(p,\boldsymbol{\theta}_j)}y(t_k) \\ \boldsymbol{\varphi}_{f,uns}(t_k) &= \frac{D(q,\boldsymbol{\eta}_{j+1})}{C(q,\boldsymbol{\eta}_{j+1})}\Bigg[\frac{A_{d,a}(q,\boldsymbol{\theta}_j)}{\overline{A}_{d,a}(q,\boldsymbol{\theta}_j)}\frac{-p}{A(p,\boldsymbol{\theta}_j)}y(t_k),\ldots,\frac{A_{d,a}(q,\boldsymbol{\theta}_j)}{\overline{A}_{d,a}(q,\boldsymbol{\theta}_j)}\frac{-p^n}{A(p,\boldsymbol{\theta}_j)}y(t_k), \\ &\qquad \frac{A_{\Delta,a}(\delta,\boldsymbol{\theta}_j)}{\overline{A}_{\Delta,a}(\delta,\boldsymbol{\theta}_j)}\frac{B_\Delta^0(\delta,\boldsymbol{\theta}_j)}{A_\Delta(\delta,\boldsymbol{\theta}_j)}\tilde{u}(t_k),\ldots,\frac{A_{\Delta,a}(\delta,\boldsymbol{\theta}_j)}{\overline{A}_{\Delta,a}(\delta,\boldsymbol{\theta}_j)}\frac{B_\Delta^m(\delta,\boldsymbol{\theta}_j)}{A_\Delta(\delta,\boldsymbol{\theta}_j)}\tilde{u}(t_k)\Bigg]^\top, \end{aligned}$$

where $B_\Delta^i(\delta,\boldsymbol{\theta}_j)/A_\Delta(\delta,\boldsymbol{\theta}_j)$ is the Delta-domain equivalent of $p^i/A(p,\boldsymbol{\theta}_j)$ for $i = 0,\ldots,m$.

Under this formulation, all the unstable poles of the estimated model at each iteration are canceled. The main drawback of this approach is that different all-pass filters are used for constructing the filtered output and filtered input signals. In other words, the model structure that is implicitly used is not (15), but rather

$$y(t_k) = \frac{\overline{A}_{d,a}(q)}{A_{d,a}(q)}\frac{A_{\Delta,a}(\delta)}{\overline{A}_{\Delta,a}(\delta)}\frac{B_\Delta(\delta)}{A_\Delta(\delta)}\tilde{u}(t_k) + \frac{\tilde{C}(q)}{\tilde{D}(q)}\varepsilon(t_k). \tag{B1}$$

However, as the sampling period tends to zero, we recover (15) since the all-pass filters in the first summand cancel out, and the delta description converges to the continuous-time one (Middleton & Goodwin, 1990).

Appendix C. Proof of Lemma 4.3

Proof. For a fixed noise model parameter vector $\boldsymbol{\eta}$, the prediction error cost can be written as

$$V(\boldsymbol{\theta}) = \frac{1}{N}\sum_{k=1}^{N}\big(y(t_k) - \hat{y}(t_k|\boldsymbol{\theta})\big)^2.$$

Its stationary points must satisfy the first-order condition $\partial V(\boldsymbol{\theta})/\partial\boldsymbol{\theta} = \mathbf{0}$, that is,

$$\frac{2}{N}\sum_{k=1}^{N}\big(y(t_k) - \hat{y}(t_k|\boldsymbol{\theta},\boldsymbol{\eta})\big)\frac{\partial}{\partial\boldsymbol{\theta}}\hat{y}(t_k|\boldsymbol{\theta},\boldsymbol{\eta}) = \mathbf{0}.$$

On the other hand, a converging point $\bar{\boldsymbol{\theta}}$ of the recursive estimator in (25) satisfies

$$\bar{\boldsymbol{\theta}} = \left[\sum_{k=1}^{N}\hat{\boldsymbol{\varphi}}_{f,uns}(t_k,\bar{\boldsymbol{\theta}})\boldsymbol{\varphi}_{f,uns}^{\top}(t_k,\bar{\boldsymbol{\theta}})\right]^{-1}\left[\sum_{k=1}^{N}\hat{\boldsymbol{\varphi}}_{f,uns}(t_k,\bar{\boldsymbol{\theta}})y_{f,uns}(t_k,\bar{\boldsymbol{\theta}})\right]. \tag{C1}$$

In the expression above, the dependence on $\bar{\boldsymbol{\theta}}$ in the instrument, regressor and filter output has been made explicit. Provided that the matrix inverse exists, (C1) can be written as

$$\sum_{k=1}^{N}\hat{\boldsymbol{\varphi}}_{f,uns}(t_k,\bar{\boldsymbol{\theta}})\left(y_{f,uns}(t_k,\bar{\boldsymbol{\theta}}) - \boldsymbol{\varphi}_{f,uns}^{\top}(t_k,\bar{\boldsymbol{\theta}})\bar{\boldsymbol{\theta}}\right) = \mathbf{0}.$$

Now, note that

$$\begin{aligned}
&y_{f,uns}(t_k,\bar{\boldsymbol{\theta}}) - \boldsymbol{\varphi}_{f,uns}^{\top}(t_k,\bar{\boldsymbol{\theta}})\bar{\boldsymbol{\theta}}\\
&\quad=\frac{\tilde{D}(q,\boldsymbol{\eta},\bar{\boldsymbol{\theta}})}{\tilde{C}(q,\boldsymbol{\eta},\bar{\boldsymbol{\theta}})}\frac{1+\bar{a}_1p+\ldots+\bar{a}_np^n}{\bar{A}(p)}y(t_k) - \frac{\tilde{D}(q,\boldsymbol{\eta},\bar{\boldsymbol{\theta}})}{\tilde{C}(q,\boldsymbol{\eta},\bar{\boldsymbol{\theta}})}\left\{\frac{\bar{b}_0+\bar{b}_1p+\ldots+\bar{b}_mp^m}{\bar{A}(p)}u(t)\right\}_{t=t_k}\\
&\quad=\frac{\tilde{D}(q,\boldsymbol{\eta},\bar{\boldsymbol{\theta}})}{\tilde{C}(q,\boldsymbol{\eta},\bar{\boldsymbol{\theta}})}\frac{A(p,\bar{\boldsymbol{\theta}})}{A(p,\bar{\boldsymbol{\theta}})}y(t_k) - \frac{\tilde{D}(q,\boldsymbol{\eta},\bar{\boldsymbol{\theta}})}{\tilde{C}(q,\boldsymbol{\eta},\bar{\boldsymbol{\theta}})}\left\{\frac{\bar{B}(p)}{\bar{A}(p)}u(t)\right\}_{t=t_k}\\
&\quad=\frac{\tilde{D}(q,\boldsymbol{\eta},\bar{\boldsymbol{\theta}})}{\tilde{C}(q,\boldsymbol{\eta},\bar{\boldsymbol{\theta}})}\left(y(t_k) - \left\{\frac{\bar{B}(p)}{\bar{A}(p)}u(t)\right\}_{t=t_k}\right).
\end{aligned}$$

This last expression coincides with the residual in (19), which means that the converging point of the estimator in (25) must solve

$$\frac{1}{N}\sum_{k=1}^{N}\big(y(t_k) - \hat{y}(t_k|\bar{\boldsymbol{\theta}},\boldsymbol{\eta})\big)\hat{\boldsymbol{\varphi}}_{f,uns}(t_k,\bar{\boldsymbol{\theta}}) = \mathbf{0}.$$

Thus, setting $\hat{\boldsymbol{\varphi}}_{f,uns}(t_k,\boldsymbol{\theta}) = \frac{\partial}{\partial\boldsymbol{\theta}}\hat{y}(t_k|\boldsymbol{\theta},\boldsymbol{\eta})$ leads to the desired conclusion. □

Appendix D. On computing the derivatives in (27)

The following expressions can be used for the implementation of the refined instrumental variable method in (25). Denote the numerator of the ZOH-equivalent of $B(p,\boldsymbol{\theta})/A(p,\boldsymbol{\theta})$ as $B_d(q,\boldsymbol{\theta})$. Since the noise model estimate $C(q,\boldsymbol{\eta})/D(q,\boldsymbol{\eta})$ does not depend on $\boldsymbol{\theta}$, we only need to compute derivatives of the form

$$\frac{\partial}{\partial a_l}\left\{\frac{B_d(q,\boldsymbol{\theta})}{A_{d,s}(q,\boldsymbol{\theta})\overline{A}_{d,a}(q,\boldsymbol{\theta})}\right\} = \frac{\frac{\partial B_d(q,\boldsymbol{\theta})}{\partial a_l}}{A_{d,s}(q,\boldsymbol{\theta})\overline{A}_{d,a}(q,\boldsymbol{\theta})} - \frac{B_d(q,\boldsymbol{\theta})\frac{\partial A_{d,s}(q,\boldsymbol{\theta})\overline{A}_{d,a}(q,\boldsymbol{\theta})}{\partial a_l}}{[A_{d,s}(q,\boldsymbol{\theta})\overline{A}_{d,a}(q,\boldsymbol{\theta})]^2}, \tag{D1}$$

and

$$\frac{\partial}{\partial a_l}\left\{\frac{A_{d,a}(q,\boldsymbol{\theta})}{\overline{A}_{d,a}(q,\boldsymbol{\theta})}\right\} = \frac{1}{\overline{A}_{d,a}(q,\boldsymbol{\theta})}\frac{\partial A_{d,a}(q,\boldsymbol{\theta})}{\partial a_l} - \frac{A_{d,a}(q,\boldsymbol{\theta})}{[\overline{A}_{d,a}(q,\boldsymbol{\theta})]^2}\frac{\partial \overline{A}_{d,a}(q,\boldsymbol{\theta})}{\partial a_l}.$$

The partial derivative of $A_{d,s}(q,\boldsymbol{\theta})\overline{A}_{d,a}(q,\boldsymbol{\theta})$ with respect to a_l can be computed by

$$\frac{\partial A_{d,s}(q,\boldsymbol{\theta})\overline{A}_{d,a}(q,\boldsymbol{\theta})}{\partial a_l} = \overline{A}_{d,a}(q,\boldsymbol{\theta})\frac{\partial A_{d,s}(q,\boldsymbol{\theta})}{\partial a_l} + A_{d,s}(q,\boldsymbol{\theta})\frac{\partial \overline{A}_{d,a}(q,\boldsymbol{\theta})}{\partial a_l},$$

which reduces the problem to differentiating $A_{d,s}(q,\boldsymbol{\theta})$, $\overline{A}_{d,a}(q,\boldsymbol{\theta})$ and $B_d(q,\boldsymbol{\theta})$. For $A_{d,s}(q,\boldsymbol{\theta})$, we exploit (14) and the chain rule for differentiation to obtain

$$\frac{\partial A_{d,s}(q,\boldsymbol{\theta})}{\partial a_l} = \sum_{j=1}^{n}\frac{\partial A_{d,s}(q,\boldsymbol{\theta})}{\partial a_j^d}\frac{\partial a_j^d}{\partial a_l} = \sum_{j=1}^{n}\sum_{k=0}^{n_s}\frac{\partial \tilde{a}_k^{d,s}}{\partial a_j^d}\frac{\partial a_j^d}{\partial a_l}q^{-k},$$

where $\partial a_j^d/\partial a_l$ can be obtained by differentiating (12), and

$$\frac{\partial \tilde{a}_k^{d,s}}{\partial a_j^d} = \begin{cases}\dfrac{1}{\tilde{a}_{j-k}^{d,a}}, & j-n_s \leq k \leq j\\ 0, & \text{otherwise.}\end{cases}$$

On the other hand, by using the fact that $\overline{A}_{d,a}(q,\boldsymbol{\theta}) = q^{-n_a}A_{d,a}(q^{-1},\boldsymbol{\theta})/a_{n_a}^{d,a}$, the partial derivative of this polynomial with respect to a_l is given by

$$\begin{aligned}\frac{\partial \overline{A}_{d,a}(q,\boldsymbol{\theta})}{\partial a_l} &= \frac{q^{-n_a}}{a_{n_a}^{d,a}}\frac{\partial A_{d,a}(q^{-1},\boldsymbol{\theta})}{\partial a_l} - \frac{q^{-n_a}A_{d,a}(q^{-1},\boldsymbol{\theta})}{(a_{n_a}^{d,a})^2}\frac{\partial a_{n_a}^{d,a}}{\partial a_l}\\ &= \frac{q^{-n_a}}{a_{n_a}^{d,a}}\sum_{j=1}^{n}\sum_{k=0}^{n_a}\frac{\partial \tilde{a}_k^{d,a}}{\partial a_j^d}\frac{\partial a_j^d}{\partial a_l}q^k - \frac{q^{-n_a}A_{d,a}(q^{-1},\boldsymbol{\theta})}{(a_{n_a}^{d,a})^2}\frac{\partial a_{n_a}^{d,a}}{\partial a_l},\end{aligned}$$

where

$$\frac{\partial \tilde{a}_k^{d,a}}{\partial a_j^d} = \begin{cases}\dfrac{1}{\tilde{a}_{j-k}^{d,s}}, & j-n_a \leq k \leq j\\ 0, & \text{otherwise.}\end{cases}$$

Finally, for the derivative of $B_d(q,\boldsymbol{\theta})$ in (D1), we compute this polynomial in terms of the continuous-time parameter vector $\boldsymbol{\theta}$. For simplicity only, we assume that $A(p,\boldsymbol{\theta})$ has no roots at $p=0$ and that $m<n$. We have

$$B_d(q,\boldsymbol{\theta}) = \frac{q^{-n}}{a_n}\mathbf{C}\text{adj}\big(q\mathbf{I}-\exp(\mathbf{A}h)\big)\mathbf{A}^{-1}\big(\exp(\mathbf{A}h)-\mathbf{I}\big)\mathbf{B},$$

where $\mathbf{A}$ is given by (13), and

$$\mathbf{B} = \begin{bmatrix}1, & 0, & \dots, & 0\end{bmatrix}^\top, \quad \mathbf{C} = \begin{bmatrix}0, & \dots, & 0, & b_m, & b_{m-1}, & \dots, & b_0\end{bmatrix}.$$

The equations above are explicit in the parameter a_l, and therefore allow direct differentiation of the polynomial $B_d(q, \boldsymbol{\theta})$ for $l = 1, \ldots, n$.

Appendix E. Proof of Lemma 5.1

Proof. At the converging point of the extended CLSRIVC estimator, $\bar{\boldsymbol{\theta}}$, and as the sample size tends to infinity, the stationarity assumptions imply that the sums in (25) converge to their expected values (Söderström, 1975). This leads to

$$\bar{\boldsymbol{\theta}} = \mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_{f,uns}(t_k, \bar{\boldsymbol{\theta}})\boldsymbol{\varphi}_{f,uns}^{\top}(t_k, \bar{\boldsymbol{\theta}})\right\}^{-1} \mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_{f,uns}(t_k, \bar{\boldsymbol{\theta}})y_{f,uns}(t_k, \bar{\boldsymbol{\theta}})\right\}.$$

Under the assumption that (30) is non-singular, we find that the extended CLSRIVC estimator satisfies

$$\begin{aligned}
&\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_{f,uns}(t_k, \bar{\boldsymbol{\theta}})\big(y_{f,uns}(t_k, \bar{\boldsymbol{\theta}}) - \boldsymbol{\varphi}_{f,uns}^{\top}(t_k, \bar{\boldsymbol{\theta}})\bar{\boldsymbol{\theta}}\big)\right\} = \mathbf{0} \\
\iff &\mathbb{E}\left\{\frac{A_{d,a}(q, \bar{\boldsymbol{\theta}})}{\overline{A}_{d,a}(q, \bar{\boldsymbol{\theta}})}\hat{\boldsymbol{\varphi}}_f(t_k, \bar{\boldsymbol{\theta}})\frac{A_{d,a}(q, \bar{\boldsymbol{\theta}})}{\overline{A}_{d,a}(q, \bar{\boldsymbol{\theta}})}\left(y(t_k) - \frac{\bar{B}(p)}{\bar{A}(p)}u(t_k)\right)\right\} = \mathbf{0} \\
\iff &\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k, \bar{\boldsymbol{\theta}})\left(y(t_k) - \frac{\bar{B}(p)}{\bar{A}(p)}u(t_k)\right)\right\} = \mathbf{0}, \qquad \text{(E1)}
\end{aligned}$$

where the last equality holds since the filter $A_{d,a}(q, \bar{\boldsymbol{\theta}})/\overline{A}_{d,a}(q, \bar{\boldsymbol{\theta}})$ is all-pass. Comparing this with the derivation leading to Equation (7) of González et al. (2021), we see that (E1) is the same pseudo-linear regression equation that the standard CLSRIVC method solves in the case when the estimates at each iteration are stable models. □

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