Enforcing stability through ellipsoidal inner approximations in the indirect approach for continuous-time system identification

Rodrigo A. González [*] **James S. Welsh** [**] **Cristian R. Rojas** [*]

[*] *Decision and Control Systems, KTH Royal Institute of Technology, 10044 Stockholm, Sweden (e-mails: grodrigo@kth.se, crro@kth.se).*
[**] *School of Electrical Engineering and Computing, University of Newcastle, Australia (e-mail: james.welsh@newcastle.edu.au)*

Abstract: Recently, a new indirect approach method for continuous-time system identification has been proposed that provides complete freedom on the number of poles and zeros of the linear and time-invariant continuous-time model structure. However, this procedure has reliability issues, as it may deliver unstable estimates even if the initialisation model and true system are stable. In this paper, we propose a method to overcome this problem. By generating ellipsoids that contain parameter vectors whose coefficients yield stable polynomials, we introduce a convex constraint in the indirect prediction error method formulation, and show that the proposed method enjoys optimal asymptotic properties while being robust in small and noisy data set scenarios. The effectiveness of the novel method is tested through extensive simulations.

Keywords: System identification; Continuous-time systems; Stability; Sampled data.

1. INTRODUCTION

In continuous-time system identification, the practitioner seeks to obtain a model of a continuous-time (CT) system given sampled input and output measurements. Two main directions have been developed in this field (Unbehauen and Rao, 1990): the direct approach, which consists in deriving a CT model directly from measured data; and the indirect approach, which first seeks a discrete-time (DT) model, and then transforms it into a CT equivalent model.

Historically, one of the shortcomings of the indirect approaches for continuous-time system identification has been the lack of robustness of the available methods (Garnier and Wang, 2008; Garnier and Young, 2014). This problem has been mostly due to the initialisation of the prediction error method (PEM) (Ljung, 2003, 2009). In these contributions, it was shown that provided some initialisation aspects are solved, indirect approaches can be competitive against direct approaches such as the SRIVC method (Young, 1981). More recently, the procedure introduced in González et al. (2018) has provided an alternative to SRIVC for estimating continuous-time systems with any prespecified relative degree in an indirect approach framework. This method chooses the initial estimate by the null-space fitting method for discrete-time system identification (Galrinho et al., 2018), and shows good performance in terms of fit and mean square error metrics.

However, the procedure in González et al. (2018) introduces a new problem: Even if the standard discrete-time PEM estimate is stable, by projecting its CT equivalent estimate into the proper subspace of the parameter space that yields the desired relative degree, it is possible that the resulting parameter vector describes an unstable system. In order to overcome this robustness issue, it is necessary to enforce stability in the indirect PEM method described in González et al. (2018).

In CT systems, enforcing stability as a constraint on the parameter space can be done through the Routh-Hurwitz criterion (Goodwin et al., 2001). However, the stability domain derived for polynomial orders greater than two is non-convex, which leads to difficulties in optimisation. This difficulty has been dealt with by obtaining convex bounds in an EM formulation for state-space models (Umenberger et al., 2018), or by introducing convex approximations of the stability region, like polyhedra in a robust control framework (Ackermann and Kaesbauer, 2003), or ellipsoids (Henrion et al., 2003). In particular, ellipsoidal approximations have been used for imposing stability in SRIVC in Ha and Welsh (2014), and for closed-loop control design in Datta et al. (2011).

In this paper, we propose an indirect algorithm for CT system identification that optimally enforces the desired relative degree while also enforcing stability on the estimate. For imposing stability, we obtain inner convex approximations of the stability region in the parameter space, and modify the indirect PEM estimate to include these convex sets as constraints in its optimisation step. By construction, the improved method is shown to enjoy the consistency and asymptotic efficiency of the indirect PEM method. Via simulations we quantify the robustness that is gained through the stability enforcement, and show that the proposed estimator is competitive against SRIVC.

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The paper is organised as follows. In Section 2 we describe the problem, while in Section 3 we recall the indirect PEM approach for continuous-time system identification. In Section 4, we present the tools for deriving the ellipsoids of interest, and introduce the novel indirect method that enforces stability in the estimate. Numerical experiments and results are provided in Section 5, and Section 6 concludes this paper.

2. PROBLEM FORMULATION

Consider a linear time-invariant, causal, stable, single input single output CT system of the form

$$\begin{aligned} y(t) &= G_0(p)u(t) \qquad (1)\\ &= \frac{\beta_{n-r}p^{n-r} + \beta_{n-r-1}p^{n-r-1} + \cdots + \beta_1 p + \beta_0}{p^n + \alpha_{n-1}p^{n-1} + \cdots + \alpha_1 p + \alpha_0}u(t), \end{aligned}$$

where p is the Heaviside operator, i.e., $pg(t) := \mathrm{d}g(t)/\mathrm{d}t$, and r is the relative degree of the system. The numerator and denominator polynomials of $G_0(p)$ are assumed to be coprime and hence no zero-pole cancellations occur. We denote $\boldsymbol{\theta}_c^0 := [\beta_{n-r} \; \ldots \; \beta_0 \; \alpha_{n-1} \; \ldots \; \alpha_0]^\top \in \mathbb{R}^{2n-r+1}$ as the true CT system parameter vector.

Suppose that the CT input signal is reconstructed through a zero-order hold (ZOH) device, and that the output is sampled with period h. A discrete-time noisy measurement of this signal, $\{y_m[k]\}$, is taken, see Fig. 1. Here, $\{e[k]\}_{k\in\mathbb{N}}$ is a zero-mean white noise sequence of variance σ^2.

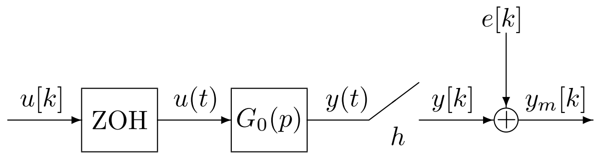

Fig. 1. System description.

Given the discrete-time input-output data measurements $\{u[k], y_m[k]\}_{k=1}^N$, sampled with period h, the goal is to obtain a CT model estimate for the system $G_0(p)$ using an indirect approach, that captures the correct relative degree and also enforces stability in the model.

Among the indirect approach methods for CT system identification, González et al. (2018) have proposed a method that imposes any predefined relative degree in the model, with success in extensive simulations. However, this procedure perturbs the poles of the PEM estimate, which may lead to instability. Thus, our goal in this paper is to ensure stability in this estimate, while preserving its statistical properties.

3. INDIRECT PEM FOR CONTINUOUS-TIME SYSTEM IDENTIFICATION

In this section we review the indirect PEM estimator for CT system identification. In contrast to the standard indirect approach, this method provides the user with freedom of choice on the number of poles and zeros of the estimated continuous-time model. In the following, we assume that the CT model structure is fixed and known [1].

[1] If the model structure of the CT system is not known, then it can be chosen through statistical measures such as the coefficient of determination or the Young Information Criterion (Young, 2011).

The standard indirect approach first computes the PEM estimator using a DT model structure of the form

$$H(q) = \frac{b_{n-1}q^{n-1} + b_{n-2}q^{n-2} + \cdots + b_1 q + b_0}{q^n + a_{n-1}q^{n-1} + \cdots + a_1 q + a_0},$$

where q is the forward-shift operator $qf[k] := f[k+1]$. We denote $\hat{\boldsymbol{\theta}}_d := [\hat{b}_{n-1} \; \ldots \; \hat{b}_0 \; \hat{a}_{n-1} \; \ldots \; \hat{a}_0]^\top \in \mathbb{R}^{2n}$ as the resulting estimate. Then, the standard method computes the continuous-time parameter vector $\hat{\boldsymbol{\theta}}_c$ by the ZOH equivalence between CT and DT systems: (using the `d2c` command in MATLAB, for example):

$$G(s) = s\mathcal{L}\left\{ \mathcal{Z}^{-1}\left\{ \frac{H(z)}{1-z^{-1}} \right\}\Big|_{k=t/h} \right\},$$

where $\mathcal{L}$ and $\mathcal{Z}$ denote the Laplace and Z transforms, respectively.

It is well known that the `d2c` conversion of a strictly proper DT system will almost surely lead to a CT equivalent of relative degree equal to one. So, instead of delivering the standard indirect approach estimate, the indirect PEM estimator for $G_0(p)$ takes $\hat{\boldsymbol{\theta}}_c$ and performs a second optimisation step, in which it locates the parameter vector that is closest to $\hat{\boldsymbol{\theta}}_c$ in an appropriate metric, subject to the constraints of relative degree.

To pose the optimisation problem of indirect PEM, the procedure requires knowledge of the covariance matrix of $\hat{\boldsymbol{\theta}}_c$. An estimate of this matrix can be obtained by noting that the parameters in $\hat{\boldsymbol{\theta}}_c$ are related to $\hat{\boldsymbol{\theta}}_d$ by the zero-order hold equivalence equations, which define a nonlinear mapping $f\colon \hat{\boldsymbol{\theta}}_c \to f(\hat{\boldsymbol{\theta}}_c) = \hat{\boldsymbol{\theta}}_d$ that is differentiable almost everywhere. Hence, the following asymptotic relationship holds for the covariance matrices of $\hat{\boldsymbol{\theta}}_d$ and $\hat{\boldsymbol{\theta}}_c$:

$$\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_d} = \mathrm{E}\{(\hat{\boldsymbol{\theta}}_d - \boldsymbol{\theta}_d^0)(\hat{\boldsymbol{\theta}}_d - \boldsymbol{\theta}_d^0)^\top\} \approx \mathbf{J}\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_c}\mathbf{J}^\top,$$

where $\boldsymbol{\theta}_d^0$ is the vector of real parameters of the discrete-time ZOH equivalent of $G_0(p)$, and $\mathbf{J}$ is the Jacobian matrix of f evaluated at a consistent estimate of $\boldsymbol{\theta}_c^0$, which can be $\hat{\boldsymbol{\theta}}_c$ or this same estimate but setting to zero the coefficients that produce an excess of relative degree.

With this covariance matrix estimate, the indirect PEM method (Söderström et al., 1991) in this context reduces to solving the following problem:

$$\begin{aligned} \tilde{\boldsymbol{\theta}}_c = \arg\min_{\boldsymbol{\theta}} (\hat{\boldsymbol{\theta}}_c - \boldsymbol{\theta})^\top \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_c}^{-1}(\hat{\boldsymbol{\theta}}_c - \boldsymbol{\theta}) \qquad (2)\\ \text{s.t.} \quad [\mathbf{I}_{r-1} \; \mathbf{0}]\,\boldsymbol{\theta} = \mathbf{0}, \end{aligned}$$

where $\mathbf{I}_{r-1}$ is the identity matrix of dimension $r-1$, $\mathbf{0}$ is the null matrix (or vector) of appropriate dimensions, and $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_c}^{-1} = \mathbf{J}^\top \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_d}^{-1}\mathbf{J}$. The optimisation problem in (2) has an explicit solution (González et al., 2018), which is given by

$$\tilde{\boldsymbol{\theta}}_c = \mathbf{C}\begin{bmatrix} \mathbf{0}_{r-1} & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{I}_{2n-r+1} \end{bmatrix}\mathbf{C}^{-1}\hat{\boldsymbol{\theta}}_c. \qquad (3)$$

where $\mathbf{C}$ is the Cholesky factorization matrix of $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_c}$ (Horn and Johnson, 2012) (i.e., a lower triangular matrix with positive diagonal entries such that $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_c} = \mathbf{C}\mathbf{C}^\top$).

The estimator (3) can be seen as the $\mathcal{L}^2$ best approximation to the PEM CT estimate $\hat{\boldsymbol{\theta}}_c$ that imposes the desired relative degree. Note that it relies on PEM giving a good initial estimate of the CT model parameters.

3.1 *Properties of the indirect PEM for CT systems*

We briefly present some properties of estimator (3), all of which are proven in González et al. (2018).

Theorem 3.1. Consider the system described by Fig. 1 and (1), where $\{e[k]\}_{k=1}^{N}$ is a Gaussian white noise sequence. Assume that the sampling frequency $2\pi/h$ is larger than twice the largest imaginary part of the s-domain poles and that there is no delay in the real system. Then, the estimator (3) is a consistent and asymptotically efficient estimator of the real vector parameter $\boldsymbol{\theta}_c^0$, provided that the DT model set (with the chosen relative degree) contains the real system.

Theorem 3.2. The asymptotic covariances of the standard indirect approach and indirect PEM with relative degree enforcement satisfy the following properties:

$\mathrm{AsCov}(\tilde{\boldsymbol{\theta}}_c - \boldsymbol{\theta}_c^0, \hat{\boldsymbol{\theta}}_c - \tilde{\boldsymbol{\theta}}_c) = \mathbf{0}$,

$\mathrm{AsCov}(\tilde{\boldsymbol{\theta}}_c - \boldsymbol{\theta}_c^0) = \mathrm{AsCov}(\hat{\boldsymbol{\theta}}_c - \boldsymbol{\theta}_c^0) - \mathrm{AsCov}(\hat{\boldsymbol{\theta}}_c - \tilde{\boldsymbol{\theta}}_c)$,

where $\mathrm{AsCov}\{\cdot\}$ denotes the asymptotic covariance of a stochastic process (Ljung, 1999).

Although the indirect PEM has strong asymptotic statistical properties, when only a small number of data points is obtained, or when the signal to noise ratio is low, the high order numerator coefficients in the standard indirect estimate can be far from zero, which produces strong perturbations in the denominator coefficients of the indirect PEM estimate. This can lead to instability, even if the standard indirect estimate is stable. To enforce stability in the model, while preserving the asymptotic properties in Theorems 3.1 and 3.2, we derive a novel indirect-PEM-based method, which is described next.

4. ENSURING STABILITY IN INDIRECT PEM

The key idea behind our new approach is to generate a closed convex stability domain in the space of the parameter coefficients, and to project the standard indirect approach PEM estimate into the intersection of this domain with the subspace that yields the correct relative degree.

Before presenting the novel indirect PEM algorithm, we introduce the techniques used to generate the closed convex stability domain. Let

$$\mathcal{D} = \{s \in \mathbb{C} \colon a + b(s+\bar{s}) + cs\bar{s} < 0\} \tag{4}$$

be a given region in the complex plane, where $a, b, c \in \mathbb{R}$. We define the vectors $\mathbf{x} := [x_0 \ x_1 \ \ldots \ x_{n-1}]^\top$ and $\bar{\mathbf{x}} := [x_0 \ x_1 \ \ldots \ x_{n-1} \ x_n]^\top$ to be the coefficients of the polynomial $x(s) = x_0 + x_1 s + \cdots + x_n s^n$, where we assume without loss of generality that $x_n = 1$. The following well-known result relates the location of the roots of $x(s)$ with a positive-definiteness condition, and is an extension of Hermite's stability criterion (Parks and Hahn, 1993).

Lemma 4.1. (Lev-Ari et al. (1991); Henrion et al. (2003)) The roots of the polynomial $x(s)$ lie in $\mathcal{D}$ if and only if

$$\mathbf{H}(\bar{\mathbf{x}}) = \sum_{i,j=0}^{n} x_i x_j \mathbf{H}_{ij} \succ 0,$$

where $\mathbf{H}_{ij} = \mathbf{H}_{ji}^\top \in \mathbb{R}^{n\times n}$ are given constant matrices depending only on $\mathcal{D}$, which are computed by solving

$$\bar{\mathbf{x}}\bar{\mathbf{x}}^\top - \tilde{\mathbf{x}}\tilde{\mathbf{x}}^\top = a\mathbf{R}_l^\top \mathbf{H}(\bar{\mathbf{x}})\mathbf{R}_l + b(\mathbf{R}_l^\top \mathbf{H}(\bar{\mathbf{x}})\mathbf{R}_r + \mathbf{R}_r^\top \mathbf{H}(\bar{\mathbf{x}})\mathbf{R}_l) + c\mathbf{R}_r^\top \mathbf{H}(\bar{\mathbf{x}})\mathbf{R}_r, \tag{5}$$

where $\mathbf{R}_l = [\mathbf{I}_n \ \mathbf{0}_{n\times 1}]$, $\mathbf{R}_r = [\mathbf{0}_{n\times 1} \ \mathbf{I}_n]$, and $\tilde{\mathbf{x}} \in \mathbb{R}^{n+1}$ is the vector of coefficients of the polynomial

$$\tilde{x}(s) = \left(\frac{b+cs}{\sqrt{b^2-ac}}\right)^n x\left(-\frac{a+bs}{b+cs}\right). \tag{6}$$

For the following result, we need to write this matrix as

$$\mathbf{H}(\bar{\mathbf{x}}) = (\mathbf{I}_n \otimes \bar{\mathbf{x}})^\top \overline{\mathbf{H}} (\mathbf{I}_n \otimes \bar{\mathbf{x}}), \tag{7}$$

where $\otimes$ denotes the Kronecker product, and $\overline{\mathbf{H}} \in \mathbb{R}^{n(n+1)\times n(n+1)}$ is formed by replacing (7) in (5) and matching polynomial coefficients. We now present a method for generating stable ellipsoids, firstly introduced in Henrion et al. (2003) and here stated in Theorem 4.1.

Theorem 4.1. Let $\mathcal{D}$ be a stability region with associated matrix $\overline{\mathbf{H}}$, and let $\mathbf{x}_C \in \mathbb{R}^n$ describe a n-th order monic polynomial with all its roots in $\mathcal{D}$. Solve the convex optimisation problem

$$\begin{aligned} \min_{\mathbf{P},\mathbf{G},\mathbf{D}} \quad & \mathrm{trace}(\mathbf{P}) \\ \text{s.t.} \quad & (\mathbf{D}\otimes \mathbf{I}_{n+1})\overline{\mathbf{H}} = \overline{\mathbf{H}}(\mathbf{D}\otimes \mathbf{I}_{n+1}) \\ & (\mathbf{D}\otimes \mathbf{I}_{n+1})\overline{\mathbf{H}} \succ \mathbf{I}_n \otimes \overline{\mathbf{P}} + \mathbf{G} \\ & \mathbf{D} = \mathbf{D}^\top \succ \mathbf{0} \in \mathbb{R}^{n\times n}, \end{aligned} \tag{8}$$

where

- $\overline{\mathbf{P}}$ is a symmetric block matrix which is partitioned as
$$\overline{\mathbf{P}} = \begin{bmatrix} -\mathbf{P} & \mathbf{P}\mathbf{x}_C \\ \mathbf{x}_C^\top \mathbf{P} & 1 - \mathbf{x}_C^\top \mathbf{P}\mathbf{x}_C \end{bmatrix},$$
where $\mathbf{P} \succ \mathbf{0}, \mathbf{P} \in \mathbb{R}^{n\times n}$, and
- $\mathbf{G}$ is a symmetric block matrix of the form
$$\mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{G}_{21}^\top & \cdots & \mathbf{G}_{n1}^\top \\ \mathbf{G}_{21} & \mathbf{0} & \cdots & \mathbf{G}_{n2}^\top \\ \vdots & & \ddots & \vdots \\ \mathbf{G}_{n1} & \mathbf{G}_{n2} & \cdots & \mathbf{0} \end{bmatrix},$$
where $\mathbf{G}_{ij} \in \mathbb{R}^{(n+1)\times(n+1)}$ skew-symmetric matrices.

Take $\mathbf{P} = \mathbf{P}_{\mathrm{opt}}$ as the solution of the optimisation problem stated above. Then, any vector $\mathbf{x}$ such that $(\mathbf{x} - \mathbf{x}_C)^\top \mathbf{P}_{\mathrm{opt}}(\mathbf{x} - \mathbf{x}_C) \leq 1$ parametrises a polynomial $x(s)$ with all its roots in $\mathcal{D}$.

Proof: See Henrion et al. (2003). □

By setting $a = 0$, $b = 1$ and $c = 0$, Theorem 4.1 provides an efficient procedure for computing an ellipsoid with center at $\mathbf{x}_C$ such that the vectors inside the ellipsoid render a stable polynomial. This is a convex constraint, which can be easily included in a convex optimisation problem. Note that $\mathbf{x}_C$ must describe a stable polynomial.

Since the stability region (in the parameter space) for the polynomial $\tilde{A}(p) = \tilde{a}_0 + \tilde{a}_1 p + \cdots \tilde{a}_{n-1}p^{n-1} + p^n$ is non-convex for $n > 2$ (Ackermann, 1993), we shall approximate this non-convex region by ellipsoids, and solve the minimisation problem of the indirect PEM in (2) for each convex region. Our main contribution can be resumed in the general algorithm we describe next.

Algorithm 4.1: Indirect PEM with stability guarantees

1: Input: $\{u[k], y_m[k]\}_{k=1}^N$, number of poles n, relative degree r, number of ellipsoids M
2: Compute $\hat{\boldsymbol{\theta}}_c$ and $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_c}$; the estimate of $\boldsymbol{\theta}_c^0$ by the standard indirect approach and its covariance matrix
3: Compute $\tilde{\boldsymbol{\theta}}_c$, by (3)
4: **if** $\tilde{\boldsymbol{\theta}}_c$ describes an unstable model **then**
5: Compute $\overline{\mathbf{H}}$ by Equations (5) and (7)
6: Pick $\boldsymbol{\alpha}^i \in \mathbb{R}^n$, $i = 1, \ldots, M$, all which describe stable polynomials
7: **for** $i = 1 : M$ **do**
8: For $\mathbf{x}_C = \boldsymbol{\alpha}^i$, obtain $\mathbf{P}_{\text{opt}}^i$ by solving (8)
9: Solve $\tilde{\boldsymbol{\theta}}_c^i = \arg\min_{\boldsymbol{\theta}} (\hat{\boldsymbol{\theta}}_c - \boldsymbol{\theta})^\top \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_c}^{-1} (\hat{\boldsymbol{\theta}}_c - \boldsymbol{\theta})$
$$\begin{aligned} \text{s.t.} \quad & [\mathbf{I}_{r-1} \ \mathbf{0}]\, \boldsymbol{\theta} = \mathbf{0}, \\ & (\boldsymbol{\theta} - \boldsymbol{\alpha}^i)^\top \mathbf{P}_{\text{opt}}^i (\boldsymbol{\theta} - \boldsymbol{\alpha}^i) \leq 1 \end{aligned}$$
10: **end for**
11: Compute
$$\tilde{\boldsymbol{\theta}}_c = \arg \min_{\{\tilde{\boldsymbol{\theta}}_c^i\}_{i=1}^M} \frac{1}{2} (\hat{\boldsymbol{\theta}}_c - \tilde{\boldsymbol{\theta}}_c^i)^\top \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_c}^{-1} (\hat{\boldsymbol{\theta}}_c - \tilde{\boldsymbol{\theta}}_c^i)$$
12: **end if**
13: Output: $\tilde{\boldsymbol{\theta}}_c$ and its associated model $\tilde{G}(p) = \tilde{B}(p)/\tilde{A}(p)$.

Since the system $G_0(p)$ is stable, the method proposed in Algorithm 4.1 enjoys the same asymptotic properties of the indirect PEM method described in Theorems 3.1 and 3.2. This is due to the fact that indirect PEM will provide unstable models with null probability as the number of data points tends to infinity, and thus, for large data sets, the proposed algorithm will deliver the same estimate as indirect PEM. However, it is expected that the novel method performs better than the estimator (3) for small data sets, or when the signal to noise ratio is poor.

Note that in step 6 of the algorithm, we propose to choose M stable polynomials, which correspond to the centers of the stable ellipsoids. These polynomials should be chosen such that the global optimum belongs to at least one ellipsoid. In practice, the user may choose to reflect the unstable poles obtained from $\hat{\theta}_c$, or may choose the coefficients from any other continuous-time identification method that provides a stable estimate given the data. In this case, any consistent method (such as PEM, or SRIVC (Pan et al., 2019)) is particularly recommended.

Remark 1. In the optimisation procedures $\overline{\mathbf{H}}$ must be only computed once, since it only depends on the number of poles of the model, whose structure is assumed fixed.

Remark 2. The proposed procedure can also extend to incorporate other constraints on the model's poles. For example, if it is known that the true system does not have poles with real part greater than $a^* < 0$, it is only needed to adjust the parameter a in (4), recompute $\overline{\mathbf{H}}$ by using (5), (6) and (7), and compute the ellipsoids accordingly.

5. SIMULATION STUDIES

In this section, we compare the proposed algorithm to other well-known indirect and direct CT system identification methods.

5.1 Experimental setup

The methods we compare are the indirect PEM method (González et al., 2018) (labeled PEMind), the proposed stable indirect PEM method (PEMind-s) and the simplified refined IV method for CT systems (SRIVC). The ellipsoids which yield stability were obtained through Theorem 4.1 by solving the LMI optimisation problem with the CVX package (Grant and Boyd, 2014) in MATLAB, using the SeDuMi solver. Two ellipsoids ($M = 2$) were computed for each stable estimate, with centers at the (reflected for stability, if necessary) standard PEM estimate, and the reflected indirect PEM estimate. For obtaining the SRIVC estimate, CONTSID 7.3 has been used (Garnier and Gilson, 2018) with default tuning parameters.

For the following experiments, several considerations have been taken regarding robustness of the indirect methods. First, following the suggestions in Ljung (2009), PEM was initialised with the DT equivalent of the estimate given by SRIVC. If PEM returned an unstable estimate, we simply reflected the unstable poles. If the DT estimate from PEM had a pole in the real negative axis, an n-th order approximation of the $(n + 1)$-th order CT estimate was obtained using the `balred` command in MATLAB. [2]

Each method has been tested under 500 Monte Carlo runs for each experiment. For each run, we recorded the normalised error of the estimated model $\|\hat{G} - G_0\|_2 / \|G_0\|_2$, and the fit measure
$$\text{Fit} = 100 \left(1 - \frac{\|\hat{y} - y\|_2}{\|y - \bar{y}\|_2} \right),$$
where $\hat{y}$ denotes the simulated output for validation data, and $\bar{y}$ is the mean value of the output signal y. Since we are interested in robustness, we analyse the spread of these measures. This is done through box plots and analysing the performance in the most challenging trials.

5.2 Tests on the Rao-Garnier system

We first consider the Rao-Garnier system (Rao and Garnier, 2002), which is a benchmark system for CT system identification methods. It is described by
$$G_0(p) = \frac{-6400p + 1600}{p^4 + 5p^3 + 406p^2 + 416p + 1600}.$$
This system has complex poles at $-0.5 \pm 1.94i$ and $-2 \pm 19.9i$, a non-minimum phase zero, and relative degree $r = 3$. Thus, the standard indirect approach delivers two extra zeros, which are be eliminated by the proposed method, while forcing stability on the estimate. Note that SRIVC is known to produce accurate estimates of the Rao-Garnier system, whereas the indirect methods often converge to local minima (Ljung, 2003).

A PRBS signal was used as input, with number of stages equal to nine, and the data length of the shortest interval set to three. This lead to a signal of length $N = 1533$. The sampling period was set to $h = 0.1$, which is considered a high sampling period for this system. The additive white noise was Gaussian, with variance set to be half the

[2] The problems encountered here can be solved by enforcing stability directly in the discrete-time PEM estimate, proposing a different sampling period, or different data length. These adjustments were not done, in order to simplify the experiment.

variance of the noiseless output (i.e. signal-to-noise ratio of 3[dB]).

We plotted the fit box plots for each method in Fig. 2. All fits under 0 were grouped, and the number of outliers of this kind were recorded in the lower part of the box plots. In this set of runs, PEMind returned 17 unstable estimates, all of which were denoted as outliers in the box plot. Figure 2 shows that by forcing stability in an optimal manner, PEMind-s is the most robust method against bad outliers while being comparable to the other estimators in terms of median value.

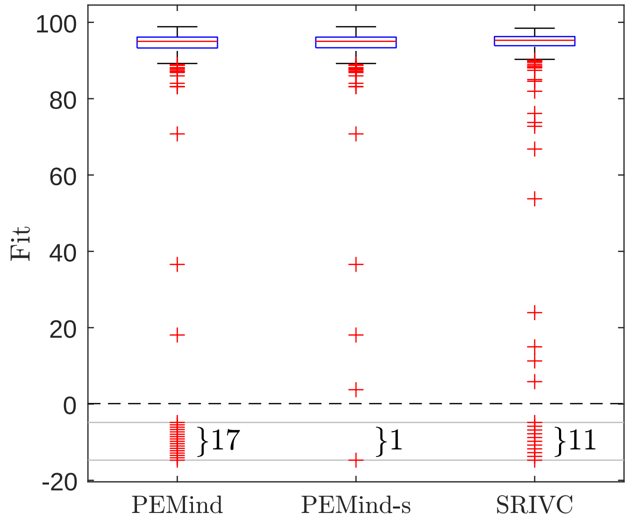

Fig. 2. Fit box plots for the Rao-Garnier system experiment. Red crosses in between the horizontal grey lines are compressed outliers (fits less than 0).

We also compared the behaviour of each estimator in the most challenging runs for each one. The worst 100 fits and normalised model errors were ordered in increasing performance (ascendant for fit, and descendant for model errors). Figure 3 shows the plots for each metric, where unstable estimates by PEMind were bounded by -100 fit, and 80 model error [3]. These plots confirm that PEMind-s is the method of choice for challenging data sets.

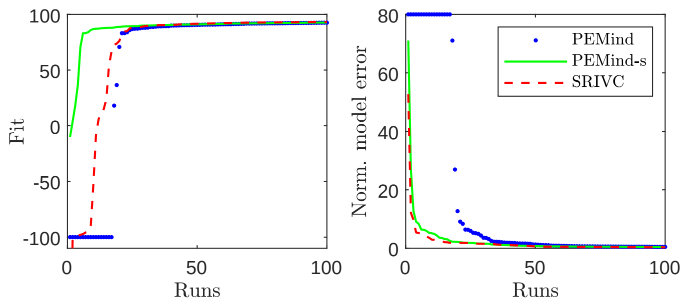

Fig. 3. Performance in each metric of the worst 100 runs per method under the Rao-Garnier system experiments.

In order to study the effect of the stability enforcement method in the poles of the system, we observed the poles of the 17 trials that returned unstable PEMind estimates. These poles are shown in Fig. 4, together with the stabilised poles obtained by PEMind-s, for the same tests. In this study, stabilisation was mostly required for the high-frequency poles, since they were poorly estimated due to the low sampling frequency. After stabilisation, the estimated poles are in fact closer to the true ones.

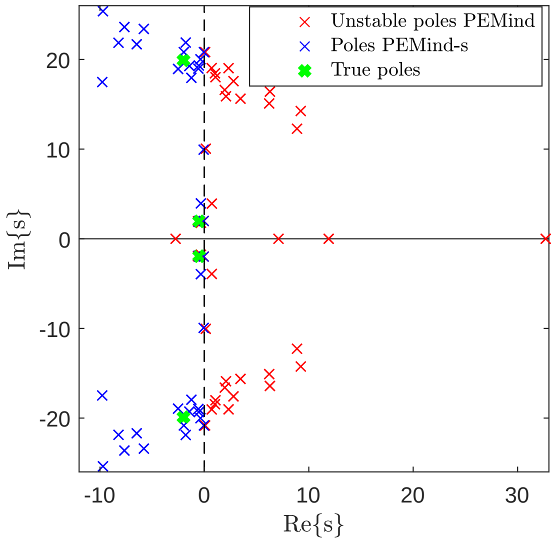

Fig. 4. Unstable poles returned by PEMind (red), stablised poles returned by PEMind-s (blue), and true poles (green).

5.3 Tests on Random systems

The proposed approach was tested against a set of 500 random systems of order 3 and relative degree 2, which was generated with the `rss` command in MATLAB. The slowest pole of each CT random system was set to have real part not larger than -0.2. The input was a unit variance Gaussian white noise of length $N = 500$, and the additive noise was also Gaussian and white, with variance such that the signal-to-noise ratio was equal to 3[dB]. The noisy output was sampled ten times faster than the fastest pole or zero of the real system.

In Fig. 5 we present box plots with the fit measure for all methods under study. In this experiment, 31 estimates by PEMind were unstable, whose fits were set to -100 in the box plot. As expected, the proposed method reduces the number of outliers of PEMind, and performs considerably better than SRIVC in terms of robustness. In addition, we have also determined the performance in the 100 most challenging runs in Figure 6. This time, unstable estimates from PEMind were chosen to have fit -100 and normalised model error equal to 15. From these plots, we find that PEMind-s is the most robust method in terms of fit and model errors.

6. CONCLUSIONS

In this paper, we have proposed a novel indirect-approach algorithm for continuous-time system identification. By introducing convex inner approximations of the stability region in the indirect PEM framework, the proposed

[3] Here it is considered that unstable systems have 2-norm equal to infinity. Thus, for plotting the measure results, we set the values for unstable models at a fixed upper bound.

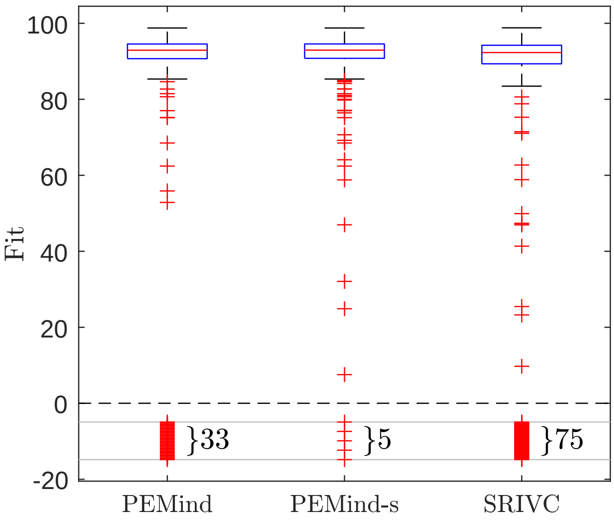

Fig. 5. Fit box plots for the set of random systems.

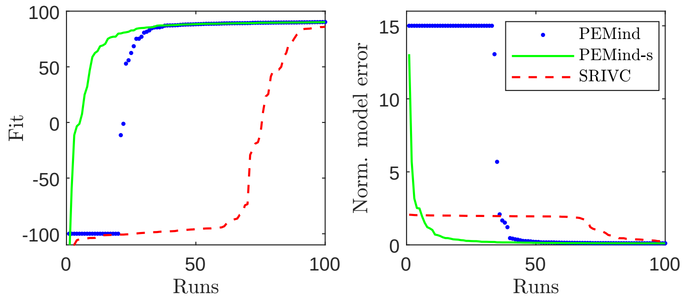

Fig. 6. Performance in each metric of the worst 100 runs per method under the set of random systems.

method guarantees the desired number of poles and zeros in the continuous-time model, while enforcing stability in the estimate. Due to its construction, it enjoys optimal asymptotic properties and it is also robust for short and noisy data set scenarios, where standard indirect PEM estimates can be highly inaccurate. Extensive simulations confirm that enforcing stability in the indirect PEM estimate is a promising approach to increasing the robustness of the indirect approach for CT system identification.

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