Stabilization of MIMO systems over additive correlated noise channels subject to multiple SNR-constraints

Rodrigo A. González, Francisco J. Vargas, and Jie Chen

Abstract— **This paper studies the mean square stabilization of MIMO discrete-time linear time-invariant systems over a MIMO additive correlated channel. We assume that such channel consists of multiple correlated SISO channels subject to independent input signal-to-noise ratio (SNR) constraints. We derive explicit conditions for which mean square stabilization can be achieved under such constraints for unstable minimum phase plants, and characterize the controller that achieves such SNR. We also present the set of admissible SNR constraints for mean square stability for a particular set of plants. Our results show that noise correlation can reduce the SNR requirements for stability compared to independent additive white noise channels. In addition, a numerical simulation is provided to illustrate the theoretical results.**

Index Terms— **Correlated noise channel, mean-square stabilization, signal-to-noise ratio constraints.**

I. Introduction

Over the last decades, control under communication constraints has received great amount of attention from the Control community [1]–[3]. This has been motivated by the accelerated integration and convergence of communications, computing, and control [4], having a wide variety of applications in the vehicle industry, teleoperations, transportation and control systems, among many other areas [5], [6].

Many studies have been held concerning fundamental limits for feedback stabilization on communication channels with constraints, and insightful results have been obtained recently on this matter for different types of constraints. Some of these works involve signal-to-noise ratio (SNR) and capacity constraints [7], quantization precision and data loss [8]–[10], time delays [11], and data-rate limits [12]. In this paper we focus on control under SNR constrained channels.

Stabilizability and performance limitations on SNR constrained additive channels (which can be viewed equivalently as power-constrained channels) were first studied on single-input single-output (SISO) plants (see e.g. [7], [13]), where it was shown that the minimum channel SNR for stabilizability depends on the plant's unstable poles, non-minimum phase zeros and the relative degree of the plant model. Linear optimal 2-degree control design under SNR constraints for SISO plants was also studied in [14]. Extensions of the mentioned results for MIMO plants have been presented in the last decade under different assumptions. In [15], additive white noise channels were considered in a two-parameter controller configuration with the assumption that the total channel input power, i.e., the sum of the input power of individual channels, is constrained. On the other hand, restrictions have been imposed on the power of each SISO channel separately in [16] in order to obtain conditions for stabilization under white noise channel scenarios. An alternative perspective is found in [17], where stabilizability conditions are studied when the channel is modeled by an input uncertainty bound. The contributions referenced above show that the stabilization and performance of MIMO plants with channel constraints depends not only on the plant's relative degree, non-minimum phase (NMP) zeros, and unstable poles, but also on their directions.

Although research on feedback systems for MIMO plants over additive channels has increased over the last decade, most of it has been held assuming uncorrelated white noise vector, as in for instance [18]–[21]. This assumption has shown to be a good starting point for interesting results to be obtained in the optimal design of controllers and scaling matrices for stabilization and performance. However, certainly more realistic models of communication links can be analyzed if noise correlations are admitted.

In this paper, we obtain explicit conditions for which a MIMO discrete-time linear time-invariant unstable minimum phase plant can be mean square stabilized over a signal-to-noise ratio constrained channel with additive correlated noise, that is, where the noise on different SISO channels is possibly affected by the same source of randomness. Also, we deduce equivalent formulations in terms of linear matrix inequalities (LMIs), and modified algebraic Riccati equations (MARE) for these stabilizability conditions. Additionally, we present the controller that achieves the minimal SNR compatible with stability. Under this analysis, the effect of noise correlation in stabilization is revealed, additional to the well known limitations given by unstable poles locations and directions, showing that the presence of such correlation can reduce the SNR requirement for stability compared with the case of non-correlated channels. Thus, our results extend the ones presented in [22] (removing pre- and post- scaling channel matrices in such setup).

The remainder of this paper is organized as follows. Section II formulates the problem. The main results, namely conditions for stability for SNR-constrained channels under correlated additive noise, are presented in Section III. Section

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IV characterizes an optimal controller for MSS under the SNR constraints. Section V illustrates the results with a numerical example and Section VI concludes this article.

Notation: Given any matrix M, its hermitian is denoted by M^H and its transpose by M^T. All the vectors and matrices involved in this paper are assumed to have compatible dimensions, and for simplicity their dimensions will generally be omitted. We denote by $\mathcal{R}$ the set of all real rational discrete-time transfer functions. We use $\mathcal{R}^{a\times b}$ to specify an element in $\mathcal{R}$ with a outputs and b inputs. The following sets are subsets of $\mathcal{R}$: $\mathcal{R}_p$ contains all proper transfer functions, $\mathcal{R}_{sp}$ contains all strictly proper transfer functions, $\mathcal{RH}_\infty$ contains all stable and proper transfer functions, $\mathcal{RH}_2$ contains all stable and strictly proper transfer functions and $\mathcal{RH}_2^\perp$ contains all the transfer functions that have no poles inside or on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. The Hermitian of a transfer matrix $H(z)$ is denoted by $H(z)^\sim$. If $H(z)$ is a transfer function with no poles on the unit circle, then the 2-norm of $H(z)$ is denoted by $\|H(z)\|_2$. For simplicity, the dependence on z is sometimes omitted.

II. Problem formulation and preliminaries

We study the following networked control architecture depicted in Fig. 1 a), where $G(z)$ is a MIMO discrete LTI system to be controlled by a proper discrete-time controller $K(z)$. The control signal is denoted by $u \in \mathbb{R}^{n_u}$.

We consider the MIMO channel described by $\tilde{y} = y + q$, where $\tilde{y} = \begin{bmatrix}\tilde{y}_1 & \tilde{y}_2 & \dots & \tilde{y}_{n_y}\end{bmatrix}^T$ is the channel output received at the controller, $y = \begin{bmatrix}y_1 & y_2 & \dots & y_{n_y}\end{bmatrix}^T$ is the system output, and $q = \begin{bmatrix}q_1 & q_2 & \dots & q_{n_y}\end{bmatrix}^T$ is a vector of stochastic additive noises, which is assumed to be a zero mean process with positive definite covariance matrix P_q. Note that P_q is not assumed to be diagonal, so q is in general composed of correlated additive noise q_i.

We assume that each channel component pair (y_i, q_i), $i = 1, 2, \dots, n_y$, is subject to a stationary SNR constraint given by

$$\gamma_i := \frac{\sigma_{y_i}^2}{\sigma_{q_i}^2} < \Gamma_i, \quad i = 1, 2, \dots, n_y, \tag{1}$$

where $\sigma_{y_i}^2$ and $\sigma_{q_i}^2$ are the variances of the i-th component of y and q respectively, γ_i is the SNR on the i-th channel and $\Gamma_i \geq 0$ is its upper limit.

Throughout this paper we will consider the following general assumptions.

Assumption 2.1:

- $G(z) \in \mathcal{R}_{sp}$ is a right-invertible, stabilizable and detectable LTI system.
- $G(z)$ is a minimum phase plant, with relative degree one, and with no poles on the unit circle. Furthermore, all the unstable poles of $G(z)$ are simple (algebraic multiplicity one). [1]
- The initial state of the plant is a second order random variable, uncorrelated with q.

[1]The assumption that $G(z)$ has only simple unstable poles is not strictly necessary, but is used in order to obtain shorter expressions in our results.

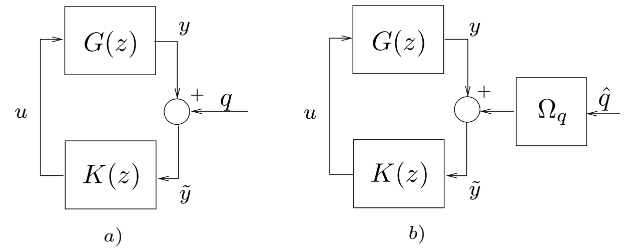

Fig. 1: Networked control system under analysis. a) With unmodeled noise q. b) Noise q modeled by Ω_q.

Our goal in this paper is to determine necessary and sufficient conditions on which the feedback system in Fig. 1.a) can be stabilized under the SNR constraints in (1). As the notion of stability, in this paper we use *mean square stability* (MSS) due to the stochastic behaviour of the signals in the loop.

Definition 2.1: The feedback system in Fig. 1.a) is said to be mean square stable (MSS) if and only if, as the time grows unbounded, the state covariance matrix of the feedback system converges to a finite matrix regardless of the initial state.

Finally, for the main results in this paper we must introduce the following technical lemma:

Lemma 2.1 ([18]): Consider transfer function matrices $\mathcal{A} \in \mathcal{RH}_\infty^{a\times b}$, $\mathcal{B} \in \mathcal{RH}_\infty^{a\times p}$, and $\mathcal{C}_i \in \mathcal{RH}_\infty^{q\times b}, i \in \{1, \dots, a\}$ with $a, b, p, q \in \mathbb{N}$. Assume that, for every $i \in \{1, \dots, a\}$, $\mathcal{B}$ and $\mathcal{C}_i$ have no zeros on the unit circle, $\mathcal{B}$ is right invertible, and $\mathcal{C}_i$ is left invertible. Denote by η_i the i-th column of the $a \times a$ identity matrix. Consider, for $i \in \{1, 2, \dots, a\}$, the minimization problems

$$J_{\inf_i} \triangleq \inf_{Q \in \mathcal{RH}_\infty} \|\eta_i^T \mathcal{A} + \eta_i^T \mathcal{B} Q \mathcal{C}_i\|_2^2,$$
$$Q_i^{\text{opt}} \triangleq \arg \inf_{Q \in \mathcal{RH}_\infty} \|\eta_i^T \mathcal{A} + \eta_i^T \mathcal{B} Q \mathcal{C}_i\|_2^2.$$

If the NMP zeros of $\mathcal{B}$ have only canonical input directions, or $\mathcal{B}$ has no NMP zeros, then there exists $Q^{\text{opt}} \in \mathcal{RH}_\infty$ such that

$$J_{\inf_i} = \|\eta_i^T \mathcal{A} + \eta_i^T \mathcal{B} Q^{\text{opt}} \mathcal{C}_i\|_2^2,$$

with

$$Q^{\text{opt}} = \sum_{i=1}^{a} Q_i^{\text{opt}},$$

and $Q_i^{\text{opt}} \in \mathcal{RH}_\infty$.

Proof: See details in [18]. ■

Lemma 2.1 shows that, in some cases, it is possible to solve a set of minimization problems using a unique optimal Youla parameter. This lemma will help us to derive our results.

III. Mean square stabilization under additive correlated noise

In this section, we will derive stabilization conditions for which the feedback loop in Fig. 1.a) is MSS. In order to do that, we notice that the processes covariances in the NCS in Fig. 1 a) are equal to those in the NCS in Fig. 1 b), where Ω_q is a positive definite matrix such that $P_q = \Omega_q \Omega_q^H$ and $\hat{q} \in \mathbb{R}^{n_y}$ is a zero-mean WSS white noise process with identity covariance matrix. With these considerations we study, without loss of generality, the architecture depicted in Fig. 1 b) instead.

Thus, it is possible to express y in terms of $\hat{q}$ as

$$\begin{aligned} y &= (I - GK)^{-1} G K \Omega_q \hat{q} \\ &=: T_{qy} \Omega_q \hat{q}. \end{aligned}$$

It is well known that for LTI systems with a second order initial state and second order WSS inputs, a controller achieves mean square stability if and only if it achieves internal stability [23]. With this knowledge in mind, we will use the Youla parametrization [24] to describe the set of all stabilizing controllers. For this purpose, we consider a doubly coprime factorization of $G(z)$ over $\mathcal{RH}_\infty$ such that $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$, where $N, M, \tilde{N}, \tilde{M} \in \mathcal{RH}_\infty$, and

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

holds for some $\tilde{X}, \tilde{Y}, X$ and Y in $\mathcal{RH}_\infty$. It is of common knowledge that any stabilizing controller $K \in \mathcal{R}_p$ can be written in terms of the double coprime factorization of $G(z)$ as $K(z) = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M})$, where the Youla parameter $Q(z)$ is in $\mathcal{RH}_\infty$. Using this parametrization, the transfer function T_{qy} can be written as follows

$$T_{qy} = N\tilde{Y} - NQ\tilde{M}. \tag{2}$$

To impose the SNR constraints on each channel, we write the variance of $y_i, i \in \{1, 2, \dots, n_y\}$ as

$$\sigma_{y_i}^2 = \|\eta_i^T T_{qy} \Omega_q\|_2^2, \tag{3}$$

where η_i^T is defined in Lemma 2.1. We also know that the variance of the i-th channel noise is given by $\sigma_{q_i}^2 = \|\eta_i^T \Omega_q\|_2^2$. So, given (1) and (3), for the system depicted in Fig. 1 b) to achieve MSS and satisfy the SNR conditions, we must have

$$\max_{i \in \{1,2,\dots,n_y\}} \min_{Q \in \mathcal{RH}_\infty} \frac{\|\eta_i^T T_{qy} \Omega_q\|_2^2}{\Gamma_i \|\eta_i^T \Omega_q\|_2^2} < 1. \tag{4}$$

The condition expressed in (4) permits us to obtain the minimum SNR that each channel must have in order to stabilize the system in Fig. 1. It is clear that the restrictions depend strongly on the factor Ω_q. Hence, understanding the effect of noise's correlation is essential for the stabilization analysis of this type of systems.

We define an all-pass filter that will enable us to obtain our results. Denote by $\mathcal{P} \triangleq \{p_1, \dots, p_{n_p}\}$ the set of unstable poles of $G(z)$. We introduce the factorization

$$\tilde{M}\Omega_q = \tilde{M}_m R, \tag{5}$$

where $\tilde{M}_m \in \mathcal{RH}_\infty$ is biproper and minimum-phase, and $R(z)$ is an all-pass filter given by

$$\begin{aligned} R &= R_{n_p} R_{n_p - 1} \cdots R_1, \\ R_k &= \left(\frac{z - p_k}{1 - p_k^* z} - 1 \right) \nu_k \nu_k^H + I, \end{aligned}$$

and ν_k is a unitary vector that can be obtained iteratively by solving, with $R_0 = I$,

$$\tilde{M}(p_k) \Omega_q R_0^{-1}(p_k) R_1^{-1}(p_k) \cdots R_{k-1}^{-1}(p_k) \nu_k = 0.$$

Given these tools, we present an explicit condition for MSS in Theorem 3.1.

Theorem 3.1: Consider the feedback systems depicted in Fig. 1, and suppose that Assumption 2.1 holds. Then, the loop in Fig. 1.a) is MSS and the SNR constraints in (1) are satisfied if and only if, for every $i \in \{1, \dots, n_y\}$,

$$\frac{1}{\sigma_{q_i}^2 \Gamma_i} \left(\eta_i^T \Omega_q R^{-1}(\infty) R^{-1}(\infty)^H \Omega_q^H \eta_i - \sigma_{q_i}^2 \right) < 1. \tag{6}$$

Proof: Using the parametrization in (2), the factorization in (5), Lemma 1 from [22], and standard 2-norm properties, we can express the argument of the minimization problem in (4) as follows

$$\begin{aligned} J_i &:= \frac{1}{\sigma_{q_i}^2 \Gamma_i} \|\eta_i^T (N\tilde{Y}\Omega_q - NQ\tilde{M}\Omega_q)\|_2^2 \\ &= \frac{1}{\sigma_{q_i}^2 \Gamma_i} \|\eta_i^T (N\tilde{Y}\Omega_q - NQ_i\tilde{M}\Omega_q)\|_2^2 \\ &= \frac{1}{\sigma_{q_i}^2 \Gamma_i} \|\eta_i^T (X\tilde{M}\Omega_q - \Omega_q - NQ_i\tilde{M}\Omega_q)\|_2^2 \\ &= \frac{1}{\sigma_{q_i}^2 \Gamma_i} \|\eta_i^T (X\tilde{M}_m - \Omega_q R^{-1} - NQ_i\tilde{M}_m)\|_2^2. \end{aligned}$$

Using the orthogonality property in 2-norm computations,

$$\begin{aligned} J_i = \frac{1}{\sigma_{q_i}^2 \Gamma_i} \Big(&\|\eta_i^T \Omega_q (R^{-1}(z) - R^{-1}(\infty))\|_2^2 \\ &+ \|\eta_i^T (X\tilde{M}_m - \Omega_q R^{-1}(\infty) - NQ_i\tilde{M}_m)\|_2^2 \Big). \end{aligned} \tag{7}$$

We can exploit the inner property of $R(z)$ in (7) to obtain

$$\begin{aligned} J_i(Q_i^{\text{opt}}) &= \frac{1}{\sigma_{q_i}^2 \Gamma_i} \Bigg(\|\eta_i^T \Omega_q R^{-1}(\infty)\|_2^2 + \|\eta_i^T \Omega_q R^{-1}(z)\|_2^2 \\ &\quad - \frac{2}{2\pi j} \oint_{z \in \mathbb{C}: |z|=1} \eta_i^T \Omega_q R^{-1}(z) R^{-1}(\infty)^H \Omega_q^{-H} \eta_i \frac{dz}{z} \Bigg) \\ &= \frac{1}{\sigma_{q_i}^2 \Gamma_i} \left(\eta_i^T \Omega_q R^{-1}(\infty) R^{-1}(\infty)^H \Omega_q^H \eta_i - \sigma_{q_i}^2 \right), \end{aligned}$$

where Q_i^{opt} is given by

$$Q_i^{\text{opt}} = N^\dagger \eta_i \eta_i^T (X\tilde{M}_m - \Omega_q R^{-1}(\infty)) \tilde{M}_m^{-1}. \tag{8}$$

Finally, (6) follows from (4). ∎

Theorem 3.1 shows that system stabilization under the SNR constraints depends strongly on the spectral factor matrix Ω_q, as well as the usual effects of unstable poles and their directions, which are present in the matrix $R(z)$. Later in this section a brief illustrative case is presented for further understanding. We also note that unlike Lemma 2 of [22], which includes scaling matrix design, this result does depend on the variance of each individual channel noise q_i.

As a corollary, in the sequel we rewrite the MSS condition in Theorem 3.1 in 3 alternative ways based on the solution of a discrete time algebraic Riccati equation (DARE), a modified algebraic Riccati equation (MARE), and a linear matrix inequality (LMI). Before stating these results, we present the following technical lemma:

Lemma 3.1: The following conditions are equivalent:

i) There exists a positive definite matrix $\mathcal{X} = \mathcal{X}^T$ such that

$$\mathcal{X} = A\mathcal{X}A^T - A\mathcal{X}C^T(P_q + C\mathcal{X}C^T)^{-1}C\mathcal{X}A^T$$
$$\gamma_i\sigma_i^2 > \eta_i^T C\mathcal{X}C^T\eta_i, \qquad i = 1, \ldots, n_y.$$

ii) There exist $\mathcal{X} = \mathcal{X}^T > 0$ and J such that

$$\mathcal{X} > (A + JC)\mathcal{X}(A + JC)^T + JP_qJ^T,$$
$$\gamma_i\sigma_i^2 > \eta_i^T C\mathcal{X}C^T\eta_i, \qquad i = 1, \ldots, n_y.$$

Proof: Omitted due to space constraints. ■

The above lemma allows us to derive the following results.

Corollary 3.1: Consider the assumptions given in Theorem 3.1. Suppose that G has a state space representation given by $(\text{diag}(A_s, A_u), [B_s^T \quad B_u^T]^T, [C_s \quad C_u], 0)$, where A_s contains all stable poles of G, A_u contains all unstable poles, and where B_s, B_u, C_s and C_u are appropriate matrices. Also, define $\Gamma \triangleq \text{diag}([\Gamma_1, \Gamma_2, \ldots, \Gamma_{n_y}])$. Then, there exists a controller $K(z)$ achieving MSS and satisfying the SNR constraints if and only if any of the following conditions are met:

i) There exists a positive definite $\mathcal{X} = \mathcal{X}^T$ such that, for every $i \in \{1, \ldots, n_y\}$

$$\eta_i^T(\Gamma P_q - C_u\mathcal{X}C_u^T)\eta_i > 0, \tag{9}$$

where $P_q = \Omega_q\Omega_q^H$ and $\mathcal{X}$ is the unique stabilizing solution of the DARE

$$\mathcal{X} = A_u\mathcal{X}A_u^T - A_u\mathcal{X}C_u^T(P_q + C_u\mathcal{X}C_u^T)^{-1}C_u\mathcal{X}A_u^T. \tag{10}$$

ii) There exists a diagonal matrix W such that $W > -(\Gamma^{-1}+I)\text{diag}(P_q)$ and such that the following MARE has a unique stabilizing solution $\mathcal{X} = \mathcal{X}^T$:

$$\mathcal{X} = A_u\mathcal{X}A_u^T - A_u\mathcal{X}C_u^T \left((\mathbf{1} + \Gamma^{-1}) \odot (P_q + C_u\mathcal{X}C_u^T) + W\right)^{-1}C_u\mathcal{X}A_u^T.$$

iii) There exist matrices $S = S^T > 0$ and V such that the following LMIs conditions are feasible:

$$\begin{bmatrix} S & SA_u + VC_u & V \\ A_u^TS + C_u^TV^T & S & 0 \\ V^T & 0 & P_q^{-1} \end{bmatrix} > 0 \tag{11}$$

$$\begin{bmatrix} \gamma_i\sigma_i^2 & \eta_i^TC_u \\ C_u^T\eta_i & S \end{bmatrix} > 0, \qquad i = \{1, \ldots, n_y\}. \tag{12}$$

Proof:

i) From the factorization in (5), we can assume without loss of generality that $R(z)$ can be calculated using a coprime factorization with co-inner denominator for the plant $G_R(z) = \Omega_q^{-1}G(z)$. Thus, we can write $G_R(z) = R^{-1}(z)\tilde{N}_R(z)$ with $R(z)$ co-inner and $\tilde{N}_R(z) \in \mathcal{RH}_\infty$. It is well known that such co-inner factor can be calculated solving a DARE (see e.g. [24]). Based on such results, we have that

$$R(\infty) = (I + \Omega_q^{-1}C_u\mathcal{X}C_u^T\Omega_q^{-H})^{-1/2}, \tag{13}$$

where $\mathcal{X} = \mathcal{X}^T > 0$ is the solution of the DARE

$$\begin{aligned} \mathcal{X} &= A_u\left[\mathcal{X} - \mathcal{X}C_u^T\Omega_q^{-H}(I + \Omega_q^{-1}C_u\mathcal{X}C_u^T\Omega_q^{-H})^{-1} \right. \\ &\qquad \left. \times\, \Omega_q^{-1}C_u\mathcal{X}\right]A_u^T \\ &= A_u\mathcal{X}A_u^T - A_u\mathcal{X}C_u^T(P_q + C_u\mathcal{X}C_u^T)^{-1}C_u\mathcal{X}A_u^T. \end{aligned}$$

Considering (13) and Theorem 3.1, the stabilization with SNR constraints problem is equivalent to checking that, for all $i \in \{1, \ldots, n_y\}$,

$$\frac{1}{\sigma_{q_i}^2\Gamma_i}\left(\eta_i^T\Omega_q(I + \Omega_q^{-1}C_u\mathcal{X}C_u^T\Omega_q^{-H})\Omega_q^H\eta_i - \sigma_{q_i}^2\right) < 1,$$

where $\mathcal{X}$ is the solution of the DARE in (10). These conditions lead to the ones written in (9) by noting that $\eta_i^T\Omega_q\Omega_q^H\eta_i = \sigma_{q_i}^2$.

ii) Given (9), there exists a diagonal matrix $\tilde{W} > 0$ such that

$$\tilde{W} - \text{diag}(P_q) + \Gamma^{-1}\text{diag}(C_u\mathcal{X}C_u^T) = 0.$$

Including this expression in the parenthesis of (10),

$$\begin{aligned} \mathcal{X} =& A_u\mathcal{X}A_u^T - A_u\mathcal{X}C_u^T(\tilde{W} - (\Gamma^{-1} + I) \odot P_q \\ &+ (\mathbf{1} + \Gamma^{-1}) \odot (P_q + C_u\mathcal{X}C_u^T))^{-1}C_u\mathcal{X}A_u^T \end{aligned}$$

If we define $W = \tilde{W} - (\Gamma^{-1} + I) \odot P_q$, we have the stated result.

iii) Now, suppose that (9) and (10) hold for some $\mathcal{X}$. Given Lemma 3.1, we conclude that there also exist J that satisfy the corresponding inequality in Lemma 3.1-ii). Applying Schur Complement condition for positive definiteness [25], we write the inequalities as follows:

$$\begin{bmatrix} \mathcal{X} & A_u + JC_u & J \\ (A_u + JC_u)^T & \mathcal{X}^{-1} & 0 \\ J^T & 0 & P_q^{-1} \end{bmatrix} > 0$$

$$\begin{bmatrix} \gamma_i\sigma_i^2 & \eta_i^TC_u \\ C_u^T\eta_i & \mathcal{X}^{-1} \end{bmatrix} > 0, \qquad i = \{1, \ldots, n_y\}.$$

Letting $S = \mathcal{X}^{-1}$ and $V = \mathcal{X}^{-1}J$, and pre- and post-multiplying the matrix

$$\begin{bmatrix} \mathcal{X}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

on both sides on the first inequality above, respectively, leads to the inequalities (11) and (12). ■

Remark 3.1: The optimization problem solved above is equivalent to minimizing the plant output power (covariance matrix) of a feedback system with a fixed channel scaling matrix given by Ω_q^{-1}, as shown in Figure 2 b). In general terms, minimizing the power of y does not imply optimal channel SNR for this alternative configuration, since the

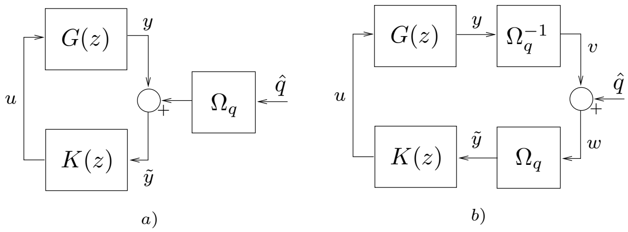

Fig. 2: Alternative interpretation a) Correlated noise architecture. b) White noise architecture, with channel scaling matrix Ω_q^{-1}.

minimization in that case should be on the power of v, not on the power of y.

In order to understand the effects of correlation, we end this section with a simple case.

Consider a plant with one pole p with direction ξ, such that $|p| > 1$. Also, the noise is such that $P_q = \Omega_q \Omega_q^H$. After some algebraic manipulation, we obtain the explicit conditions for stability given by Theorem 3.1 as

$$\max_{i \in \{1,2,\dots,n_y\}} \frac{(|p|^2 - 1)\cos^2 \angle(\eta_i, \xi)}{\sigma_{q_i}^2 \Gamma_i \|\Omega_q^{-1}\xi\|^2} < 1, \tag{14}$$

where $\cos\angle(\eta_i, \xi)$ is defined as $|\eta_i^T \xi|$.

Equation (14) states that stabilization under SNR constraints depends directly on how unstable the pole is, which is a natural restriction. Also, the pole's direction plays a crucial role, as it makes clear that if the control effort of the unstable pole is concentrated on one particular channel, this channel must have greater SNR in order to stabilize the system. This gives an insight on the relative proportions in which the SNR should be allocated in each channel. Finally, the correlation influence is made explicit. If the noise q has high correlation between each component, then Ω_q could be close to singular and the SNR requirements could drop drastically.

IV. Minimum SNR controller

We now study the form of the controller related with the minimal SNR necessary for MSS.

Lemma 4.1: Suppose that Assumption 2.1 holds, and consider the state-space notation given in 3.1. A controller $K^{\text{opt}}(z)$ that achieves the minimal SNR necessary for MSS under individual SNR constraints, is given by

$$K^{\text{opt}}(z) = G^\dagger \Omega_q (R^{-1}(\infty) - R^{-1}(z)) R(\infty) \Omega_q^{-1}, \tag{15}$$

where $G^\dagger$ is a right inverse of $G(z)$. Furthermore, if the state-space matrix C of $G(z)$ is invertible, then $K^{\text{opt}}(z)$ is a constant gain matrix given by

$$K^{\text{opt}} = -B^\dagger A \mathcal{X} C^T (\Omega_q \Omega_q^H + C\mathcal{X}C^T)^{-1},$$

where $B^\dagger$ is a pseudo-inverse of B, and where $\mathcal{X}$ is the unique stabilizing solution of the DARE

$$\mathcal{X} = A\mathcal{X}A^T - A\mathcal{X}C^T(\Omega_q\Omega_q^H + C\mathcal{X}C^T)^{-1} C\mathcal{X}A^T. \tag{16}$$

Proof: Using Lemma (2.1), from (8) we write an optimal Youla parameter Q^{opt} as

$$Q^{\text{opt}} = \sum_{i=1}^{n_y} Q_i^{\text{opt}} = N^\dagger \eta_i \eta_i^T (X\tilde{M}_m - \Omega_q R^{-1}(\infty))\tilde{M}_m^{-1}.$$

This characterization of Q^{opt} allows us to obtain an explicit expression for the optimal controller $K(Q^{\text{opt}})$. After some tedious calculation, the controller can be expressed as follows

$$\begin{aligned} K^{\text{opt}} &= (Y - MQ^{\text{opt}})(X - NQ^{\text{opt}})^{-1} \\ &= Y\tilde{M}_m R(\infty)\Omega_q^{-1} - MN^\dagger X \tilde{M}_m R(\infty)\Omega_q^{-1} + MN^\dagger \\ &= -M(N^\dagger X\tilde{M} - \tilde{Y})\Omega_q R^{-1} R(\infty)\Omega_q^{-1} + MN^\dagger. \end{aligned}$$

Note that the expression in parenthesis is a pseudo inverse of N, because

$$N(N^\dagger X\tilde{M} - \tilde{Y}) = X\tilde{M} - N\tilde{Y} = I.$$

Thus, K^{opt} must satisfy the following equality

$$GK^{\text{opt}} = \Omega_q(R^{-1}(\infty) - R^{-1}(z))R(\infty)\Omega_q^{-1}. \tag{17}$$

If we assume that K^{opt} has the form $K^{\text{opt}} := G^\dagger \tilde{K}$, where $\tilde{K}$ is an auxiliary variable, we obtain (15) directly from (17).

Without loss of generality, we can write $\Omega_q^{-1}G(z) = R^{-1}(z)\tilde{N}_R(z)$, where $R(z)$ and $\tilde{N}_R(z)$ are obtained by the state-space realizations

$$\begin{aligned} R(z) &= (A + L\Omega_q^{-1}C, \quad L, \quad Q^{-1/2}\Omega_q^{-1}C, \quad Q^{-1/2}) \\ \tilde{N}_R(z) &= (A + L\Omega_q^{-1}C, \quad B, \quad Q^{-1/2}\Omega_q^{-1}C, \quad 0), \end{aligned}$$

where $Q = I + \Omega_q^{-1}C\mathcal{X}C^T\Omega_q^{-H}$, $L = -A\mathcal{X}C^T\Omega_q^{-T}Q^{-1}$, and $\mathcal{X} = \mathcal{X}^T$ is the solution of the DARE (16). Naturally, R^{-1} has state space representation $R^{-1} = (A, LQ^{1/2}, -\Omega_q^{-1}C, Q^{1/2})$, and therefore the controller K can be expressed as

$$\begin{aligned} K^{\text{opt}} =& B^\dagger(zI - A)C^{-1}\Omega_q \\ & \times (\Omega_q^{-1}C(zI - A)^{-1}LQ^{1/2})Q^{-1/2}\Omega_q^{-1} \\ =& -B^\dagger A\mathcal{X}C^T(\Omega_q\Omega_q^T + C\mathcal{X}C^T)^{-1}, \end{aligned}$$

with $\mathcal{X}$ given by (16). That is, for this case, the optimal controller is just a constant gain matrix. ■

Lemma 4.1 presents the controller that achieves the minimal SNR compatible with MSS, which depends on the plant model and on the noise spectral factor. In the particular case when the plant matrix C is invertible, the minimum-SNR controller is a constant matrix, which is not surprising due to the fact that, in that case, the problem is essentially an state feedback control problem.

V. Numerical Example

Consider four plants given by the state space representations $G_i = (A, B, C_i, 0_{2\times 2})$, where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 2.5 & 0.5 \\ 1 & 2 \end{bmatrix},$$

$$C_i = \begin{bmatrix} 1 & \sqrt{k_i} \\ 0 & \sqrt{1 - k_i} \end{bmatrix}, k_i \in \{0, 0.3, 0.6, 0.9\}.$$

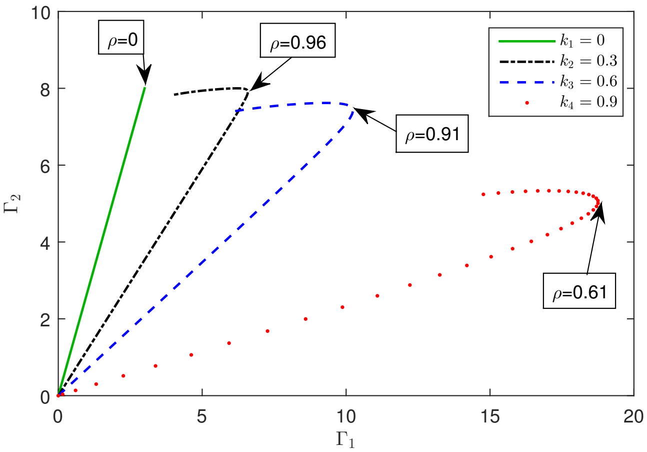

Fig. 3: Minimum SNR limits for stabilization of $G_i(z)$ for $k_1 = 0$ (green, solid), $k_2 = 0.3$ (black, dash-dot), $k_3 = 0.6$ (blue, dashed), and $k_4 = 0.9$ (red, dotted), while varying the correlation coefficient ρ from 0 to 1.

This plant has two unstable poles at $p = 2$ and $p = 3$, with directions $\xi_{1_i} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and $\xi_{2_i} = \begin{bmatrix} \sqrt{k_i} & \sqrt{1-k_i} \end{bmatrix}^T$ respectively, where $k_i \in \{0, 0.3, 0.6, 0.9\}$ is left as a parameter in order to analyze the effect of the pole direction. They do not have finite zeros, and have relative degree 1. Also, consider the covariance matrix of q as

$$P_q = \begin{bmatrix} 2 & \sqrt{6}\rho \\ \sqrt{6}\rho & 3 \end{bmatrix}, \rho \in [0, 1].$$

We now determine the minimum SNR requirements such that the systems in Fig. 1 are MSS. For every fixed k_i, we vary the coefficient ρ in order to plot the effect of correlation for different pole directions. The results are shown in Fig. 3. Each segment is plotted from no correlation (top end of each segment) to full correlation, where Ω_q is singular. For this degenerated case, all the systems can be stabilized without requirements of SNR on all channels. This result can be understood intuitively by the denominator of the conditions (14). It is seen from Fig. 3 that for G_1, the case where poles have canonical directions, noise correlation always improves the SNR requirements, while for a more general pole direction, small noise correlation will worsen the SNR requirements and high to very high noise correlation will lower the SNR requirements.

VI. Conclusions

This paper addressed the problem of mean square stabilization (MSS) of MIMO discrete-time LTI plants over additive correlated channels. Explicit conditions for MSS under individual independent SNR constraints have been obtained for unstable minimum phase plants. The direct effect of noise correlation in stabilization has been revealed, and an example exposing the correlated noise case has been put forward. Further research will be held concerning presence of channel scaling matrices in order to improve the theoretical SNR limits under this configuration.

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