The SRIVC algorithm for continuous-time system identification with arbitrary input excitation in open and closed loop

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***Abstract*— Continuous-time system identification has primarily dealt with sampled input and output data for constructing continuous-time models. However, sampled signals can lead to inaccurate models if their intersample behavior is not addressed appropriately. In this paper, this effect is explored in detail with respect to the SRIVC and CLSRIVC estimators, which are some of the most popular methods for open and closed-loop continuous-time system identification respectively. Based on our consistency analysis, we propose an algorithm that alleviates the asymptotic bias of these methods for arbitrary input excitations and provide an alternative procedure to achieve consistent estimates for band-limited signals. Simulation examples show the effectiveness of our approach.**

***Index Terms*— System identification; Continuous-time systems; Parameter estimation; Instrumental variables.**

I. Introduction

System identification deals with the problem of constructing models of systems based on input and output data. Additionally to the historical interest in discrete-time models, continuous-time system identification has been re-visited in depth in the recent years [1] due to its advantages over its discrete-time counterpart in, e.g., robustness, physical insights, and handling of irregular sampling [2].

In most continuous-time system identification methods, the derivatives of the input and output signals are implicitly or explicitly needed. A common way to compute these derivatives is via prefiltering the signals with state variable filters, which has ultimately led to the Simplified Refined Instrumental Variable method for Continuous-time systems (SRIVC, [3]). The SRIVC method is one of the most popular methods available for continuous-time system identification [4], and has been extended to handle time-delays [5], multi-input single-output [6], and closed-loop systems [1, Chapter 5], with its closed-loop variant known as CLSRIVC.

In the case of zero or first-order hold (ZOH or FOH [7], respectively) input signals, the SRIVC estimator has recently been proven to be generically consistent [8] and asymptotically efficient [9]. However, this estimator is no longer consistent if there is a misspecification of the intersample behavior of the input signal when forming the regressor vector, shown in [8]. The loss in consistency has been suggested previously [10], although the remedy to this problem, based on a comprehensive statistical analysis of the algorithm, has only been explored recently and solely for continuous-time multisine input signals [11]. The misspecification of the intersample behaviors of the signals is also significant for closed-loop system identification using the CLSRIVC method, as this estimator only requires samples of the system input and reference but does not consider that the underlying continuous-time signals usually cannot be described with hold devices. This phenomenon has been somewhat overlooked in the literature, as in previous contributions the controller has been assumed to be fully digital [12] leading to ZOH inputs, or the input and output have been sampled at very high rates [13], which in practice can mitigate the consistency issues to a certain degree. Note that high sampling rates and input-output time-synchronism cannot always be achieved in practical applications.

In summary, the main results of this paper are:

- We characterize the asymptotic bias of the SRIVC and CLSRIVC methods for when the intersample behavior of the input is misspecified. With this, we prove that these procedures are asymptotically biased due to a misconstruction of the regressor vector.
- We propose a general procedure, based on oversampling, to compute the regressor and instrument vectors of the SRIVC and CLSRIVC methods that leads to alleviating the bias for arbitrary input signals. An alternative filtering procedure is also presented for band-limited input or reference excitation.
- The theoretical findings and methods are illustrated via simulation examples.

The rest of the paper is structured as follows. In Section II the setup for open and closed-loop system identification is described. Section III introduces the SRIVC and CLSRIVC estimators, and their consistency issues are exposed. In Section IV we propose fixes to these algorithms that lead to consistent estimators in open and closed-loop. Simulation examples that illustrate the methods are presented in Section V, and conclusions are drawn in Section VI.

II. Problem formulation

Consider a linear time-invariant, causal, single-input single-output, continuous-time system

$$x(t) = \frac{B^*(p)}{A^*(p)} u(t)$$

where p is the Heaviside operator, i.e., $pg(t) = \mathrm{d}g(t)/\mathrm{d}t$. The system numerator and denominator polynomials are assumed

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to be coprime with orders m^* and n^* respectively, i.e.,

$$B^*(p) = b^*_{m^*}p^{m^*} + b^*_{m^*-1}p^{m^*-1} + \cdots + b^*_1 p + b^*_0$$
$$A^*(p) = a^*_{n^*}p^{n^*} + a^*_{n^*-1}p^{n^*-1} + \cdots + a^*_1 p + 1, \quad (1)$$

and the system parameter vector is described by

$$\boldsymbol{\theta}^* = \left[a^*_1,\ a^*_2,\ \ldots,\ a^*_{n^*},\ b^*_0,\ b^*_1,\ \ldots,\ b^*_{m^*}\right].$$

In this work, we make the key assumption that the intersample behavior of the system input is known exactly between consecutive samples, and it is arbitrary. That is, the input is not only allowed to be generated through a hold device such as a ZOH or FOH, but can also be a continuous-time band-limited signal or, in general, any continuous-time signal. By discarding the common assumption of the input being interpolated through a hold device, our work covers the identification of cascaded continuous-time systems, as well as continuous-time closed-loop systems without a sampler between controller and plant.

In this paper we study both open and closed loop system identification. For the open loop case, we suppose that the output signal is regularly sampled at time instants $\{t_k\}_{k=1}^N$ and the resulting output is contaminated by an additive zero-mean stochastic process $v(t_k)$. That is,

$$y(t_k) = x(t_k) + v(t_k), \quad k = 1, 2, \ldots, N.$$

On the other hand, the closed loop framework we consider is portrayed in Figure 1. Here, the continuous-time controller $C(p) = P(p)/L(p)$ is assumed known, and $r(t)$ is a known arbitrary continuous-time reference signal which is assumed to be independent of the noise disturbance $v(t)$. The continuous-time input and output are described by

$$y(t) = \frac{G^*(p)C(p)}{1+G^*(p)C(p)}r(t) + \frac{1}{1+G^*(p)C(p)}v(t),$$
$$u(t) = \frac{C(p)}{1+G^*(p)C(p)}r(t) - \frac{C(p)}{1+G^*(p)C(p)}v(t).$$

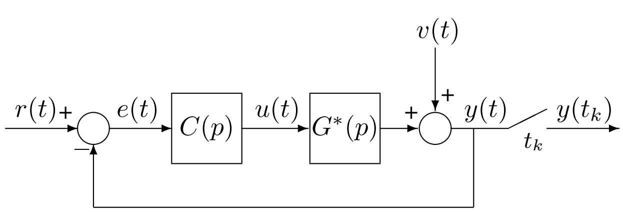

Fig. 1. Block diagram for the closed-loop framework.

The goal is to obtain a continuous-time transfer function estimate for $G^*(p) := B^*(p)/A^*(p)$ based on data. For the open-loop case we assume knowledge of N output data measurements $\{y(t_k)\}_{k=1}^N$ the continuous-time signal $\{u(t)\}_{t\in[t_1,t_N]}$, and for the closed-loop framework we also assume access to $\{r(t)\}_{t\in[t_1,t_N]}$. Contrary to most past contributions on this topic [12], [14], our setup does not include a sampler after the controller. This implies that the signal that is fed to the plant in general cannot be represented as a sampled signal passed through a ZOH or FOH device.

If the SRIVC or CLSRIVC estimators are used for estimating $\boldsymbol{\theta}^*$, we will show that in many cases an asymptotic bias will be present due to a misspecification of the intersample behavior of the input in the regressor vector of the methods. Our focus will be to extend the applicability of these estimators for arbitrary input excitations.

III. The SRIVC and CLSRIVC estimators

First introduced in [3], the SRIVC estimator for continuous-time systems in open-loop is an iterative instrumental variables algorithm that generates parameter-dependent filters that are applied to sampled input and output data at each iteration. The closed-loop version of this algorithm, known as CLSRIVC, has been studied extensively in [1, Chapter 5] and is constructed similarly to its open-loop variant, as seen next.

The SRIVC and CLSRIVC procedures require updating, at each iteration, a filtered regressor vector $\boldsymbol{\varphi}_f(t_k, \boldsymbol{\theta}_j)$, a filtered instrument vector $\hat{\boldsymbol{\varphi}}_f(t_k, \boldsymbol{\theta}_j)$, and a filtered output $y_f(t_k, \boldsymbol{\theta}_j)$. These vectors are formed by the derivatives of filtered versions of the input and output data, therefore avoiding direct computation of the derivatives of sampled signals. The filtering process is implemented in discrete-time based on a ZOH or FOH intersample behavior assumption on the sampled signals. For the open loop case, these are given by

$$\boldsymbol{\varphi}_f(t_k, \boldsymbol{\theta}_j) = \left[\frac{-p}{A_j(p)}y(t_k),\ \ldots,\ \frac{-p^n}{A_j(p)}y(t_k),\right.$$
$$\left.\frac{1}{A_j(p)}u(t_k),\ \ldots,\ \frac{p^m}{A_j(p)}u(t_k)\right]^\top, \quad (2)$$
$$\hat{\boldsymbol{\varphi}}_f(t_k, \boldsymbol{\theta}_j) = \left[\frac{-pB_j(p)}{A_j^2(p)}u(t_k), \ldots, \frac{-p^nB_j(p)}{A_j^2(p)}u(t_k),\right.$$
$$\left.\frac{1}{A_j(p)}u(t_k),\ \ldots,\ \frac{p^m}{A_j(p)}u(t_k)\right]^\top, \quad (3)$$
$$y_f(t_k, \boldsymbol{\theta}_j) = \frac{1}{A_j(p)}y(t_k), \quad (4)$$

where $A_j(p)$ and $B_j(p)$ are the denominator and numerator polynomials of the j-th iterate of the SRIVC method (i.e., associated with $\boldsymbol{\theta}_j$).

Remark 3.1: In this paper, the notation $G(p)x(t_k)$ means that the sampled signal $\{x(t_k)\}_{k=1}^N$ is being interpolated via a ZOH or FOH device, filtered through the continuous-time filter $G(p)$, and later evaluated at $t = t_k$. In contrast, the notation $\{G(p)x(t)\}_{t=t_k}$ (or with square brackets for the vector case) means that the continuous-time signal $x(t)$ is filtered through $G(p)$ and later evaluated at $t = t_k$.

As seen in (3), the filtered instrument vector recreates a filtered version of the noise-less output and input. This principle is extended for the closed-loop variant, in which the instrument vector resembles the filtered output and input signals present in the regressor in (2), but without explicit dependence on the noise $v(t)$. For notation purposes, we introduce the nominal complementary and control sensitivity functions as

$$T_{o,j}(p) := \frac{G_j(p)C(p)}{1+G_j(p)C(p)}; \quad S_{uo,j}(p) := \frac{C(p)}{1+G_j(p)C(p)}$$

respectively. The filtered instrument vector for the closed-loop case is then given by

$$\hat{\boldsymbol{\varphi}}_f(t_k,\boldsymbol{\theta}_j) = \left[\frac{-pT_{o,j}(p)}{A_j(p)}r(t_k), \ldots, \frac{-p^nT_{o,j}(p)}{A_j(p)}r(t_k), \frac{S_{uo,j}(p)}{A_j(p)}r(t_k), \ldots, \frac{p^m S_{uo,j}(p)}{A_j(p)}r(t_k)\right]^\top. \quad (5)$$

Algorithm 1: SRIVC and CLSRIVC

Input: Model order (n,m), initial vector estimate $\boldsymbol{\theta}_1 \in \mathbb{R}^{n+m+1}$, tolerance ϵ, maximum number of iterations `MaxIter`, $\{u(t_k), y(t_k)\}_{k=1}^N$ for open loop, $\{r(t_k), u(t_k), y(t_k)\}_{k=1}^N$ for closed loop
Using $\boldsymbol{\theta}_1$, form the model polynomials $A_1(p)$ and $B_1(p)$
$j \leftarrow 1$, flag $\leftarrow 1$
while flag $= 1$ and $j \leq$ `MaxIter` **do**
  if system is operating in open loop **then**
    Prefilter $\{u(t_k), y(t_k)\}_{k=1}^N$ to form $\boldsymbol{\varphi}_f(t_k,\boldsymbol{\theta}_j)$, $\hat{\boldsymbol{\varphi}}_f(t_k,\boldsymbol{\theta}_j)$ and $y_f(t_k,\boldsymbol{\theta}_j)$ by (2), (3) and (4)
  end if
  if system is operating in closed loop **then**
    Prefilter $\{r(t_k), u(t_k), y(t_k)\}_{k=1}^N$ to form $\boldsymbol{\varphi}_f(t_k,\boldsymbol{\theta}_j)$, $\hat{\boldsymbol{\varphi}}_f(t_k,\boldsymbol{\theta}_j)$ and $y_f(t_k,\boldsymbol{\theta}_j)$ by (2), (5) and (4)
  end if
  Compute the parameter estimate

$$\boldsymbol{\theta}_{j+1} \leftarrow \left[\sum_{k=1}^N \hat{\boldsymbol{\varphi}}_f(t_k,\boldsymbol{\theta}_j)\boldsymbol{\varphi}_f^\top(t_k,\boldsymbol{\theta}_j)\right]^{-1}\left[\sum_{k=1}^N \hat{\boldsymbol{\varphi}}_f(t_k,\boldsymbol{\theta}_j)y_f(t_k,\boldsymbol{\theta}_j)\right]$$

  if $A_{j+1}^{-1}(p)$ (or $T_{o,j+1}(p)$) is unstable **then**
    Reflect the unstable poles of $A_{j+1}^{-1}(p)$ (or $T_{o,j+1}(p)$) into the stable region of the complex plane
  end if
  if $\frac{\|\boldsymbol{\theta}_{j+1}-\boldsymbol{\theta}_j\|_2}{\|\boldsymbol{\theta}_j\|_2} < \epsilon$ **then**
    flag $\leftarrow 0$
  end if
  $j \leftarrow j+1$
end while
Output: $\boldsymbol{\theta}_j$ and its associated model $B_j(p)/A_j(p)$.

The main shortcoming of the SRIVC and CLSRIVC estimators, which are jointly described in Algorithm 1, is their loss of consistency when there is a misspecification of the intersample behavior of the sampled input signal. Here we will analyze this effect for both algorithms jointly.

For any fixed N, as the number of iterations j tend to infinity, the converging point $\bar{\boldsymbol{\theta}}$ of these estimators satisfies

$$\left[\frac{1}{N}\sum_{k=1}^N \hat{\boldsymbol{\varphi}}_f(t_k,\bar{\boldsymbol{\theta}})\boldsymbol{\varphi}_f^\top(t_k,\bar{\boldsymbol{\theta}})\right]^{-1} \times \left[\frac{1}{N}\sum_{k=1}^N \hat{\boldsymbol{\varphi}}_f(t_k,\bar{\boldsymbol{\theta}})\Big(y_f(t_k,\bar{\boldsymbol{\theta}}) - \boldsymbol{\varphi}_f^\top(t_k,\bar{\boldsymbol{\theta}})\bar{\boldsymbol{\theta}}\Big)\right] = \mathbf{0}. \quad (6)$$

Note that

$$y_f(t_k,\bar{\boldsymbol{\theta}}) - \boldsymbol{\varphi}_f^\top(t_k,\bar{\boldsymbol{\theta}})\bar{\boldsymbol{\theta}} = y(t_k) - \frac{\bar{B}(p)}{\bar{A}(p)}u(t_k),$$

where $\bar{A}(p)$ and $\bar{B}(p)$ are the A and B polynomials evaluated with coefficients described in $\bar{\boldsymbol{\theta}}$. On the other hand, due to the fact that the input (or reference, in the closed loop case) is uncorrelated with the noise present in $y(t_k)$, we have

$$\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k,\bar{\boldsymbol{\theta}})y(t_k)\right\} = \mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k,\bar{\boldsymbol{\theta}})\left\{\frac{B^*(p)}{A^*(p)}u(t)\right\}_{t=t_k}\right\},$$

where we have used the notation introduced in Remark 3.1. As the sample size tends to infinity and under mild assumptions, the sums in (6) converge to their expected values [15], thus leading to

$$\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k,\bar{\boldsymbol{\theta}})\boldsymbol{\varphi}_f^\top(t_k,\bar{\boldsymbol{\theta}})\right\}^{-1} \times \mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k,\bar{\boldsymbol{\theta}})\left(\left\{\frac{B^*(p)}{A^*(p)}u(t)\right\}_{t=t_k} - \frac{\bar{B}(p)}{\bar{A}(p)}u(t_k)\right)\right\} = \mathbf{0}.$$

Provided some identifiability conditions concerning the model structure and persistence of excitation are met, we can show that $\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k,\bar{\boldsymbol{\theta}})\boldsymbol{\varphi}_f^\top(t_k,\bar{\boldsymbol{\theta}})\right\}$ is generically non-singular[1], which leads to

$$\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k,\bar{\boldsymbol{\theta}})\left(\left\{\frac{B^*(p)}{A^*(p)}u(t)\right\}_{t=t_k} - \frac{\bar{B}(p)}{\bar{A}(p)}u(t_k)\right)\right\} = \mathbf{0}. \quad (7)$$

This equation characterizes the asymptotic bias of the SRIVC and CLSRIVC estimators when the intersample behavior of the system input does not match the one used in the algorithms. For the sequel, we will consider the open-loop case (i.e., $\hat{\boldsymbol{\varphi}}_f(t_k,\bar{\boldsymbol{\theta}})$ as in (3)), although the same can be concluded about the CLSRIVC method after a similar derivation. The filtered instrument $\hat{\boldsymbol{\varphi}}_f(t_k,\bar{\boldsymbol{\theta}})$ can be computed as

$$\hat{\boldsymbol{\varphi}}_f(t_k,\bar{\boldsymbol{\theta}}) = \mathbf{S}(-\bar{B},\bar{A})\frac{1}{\bar{A}^2(p)}\mathbf{u}_d(t_k),$$

where $\mathbf{S}(-\bar{B},\bar{A})$ is the Sylvester matrix formed by the polynomials $-\bar{B}(p)$ and $\bar{A}(p)$, which is non-singular if $\bar{B}(p)$ and $\bar{A}(p)$ are coprime [17, Lemma A3.1], and $\mathbf{u}_d(t_k)$ is a vector formed by the derivatives of the input, i.e.,

$$\mathbf{u}_d(t_k) = [p^{n+m}u(t_k),\, p^{n+m-1}u(t_k),\, \ldots,\, u(t_k)]^\top.$$

Also, we can write

$$\frac{\bar{B}(p)}{\bar{A}(p)}u(t_k) - \frac{B^*(p)}{A^*(p)}u(t_k) = \frac{1}{\bar{A}(p)A^*(p)}\mathbf{u}_d^\top(t_k)\mathbf{h},$$

where $\mathbf{h}$ is a vector that contains the coefficients of $A^*(p)\bar{B}(p) - \bar{A}(p)B^*(p)$. Thus, (7) is equivalent to

$$\mathbb{E}\left\{\frac{1}{\bar{A}^2(p)}\mathbf{u}_d(t_k)\frac{1}{\bar{A}(p)A^*(p)}\mathbf{u}_d^\top(t_k)\right\}\mathbf{h} = \mathbb{E}\left\{\frac{1}{\bar{A}^2(p)}\mathbf{u}_d(t_k)\Delta(t_k)\right\}, \quad (8)$$

where we have defined the error signal $\Delta(t_k)$ as

$$\Delta(t_k) := \left\{\frac{B^*(p)}{A^*(p)}u(t)\right\}_{t=t_k} - \frac{B^*(p)}{A^*(p)}u(t_k).$$

In the case where the input is exactly reconstructed by a known hold device, we have $\Delta(t_k) = 0$ and thus $\mathbf{h} = \mathbf{0}$,

[1] The complete result, assumptions and analysis for the open-loop case can be found in [8], whereas the closed-loop case can be found in [16].

since the matrix on the left hand side of (8) is known to be generically non-singular by Theorem 1 of [8]. So, we reach

$$\mathbf{h} = \mathbf{0} \iff \frac{\bar{B}(p)}{\bar{A}(p)} = \frac{B^*(p)}{A^*(p)},$$

i.e., the SRIVC estimator is generically consistent. Otherwise, if the intersample behavior of the input is misspecified in the computation of the regressor, then $\Delta(t_k) \neq 0$ and the expectation on the right hand side of (8) is in general different from zero. This leads to $\mathbf{h} \neq \mathbf{0}$, which means that the SRIVC estimator will be asymptotically biased.

In the next section, we introduce algorithms that deal with this problem.

IV. SRIVC and CLSRIVC for arbitrary signals

The asymptotic bias of SRIVC and CLSRIVC that has been characterized in the previous section can be canceled if $\Delta(t_k) = 0$. This is achieved if the intersample behavior of the input in the filtered regressor in (2) matches that of the input of the true system, i.e., if

$$\boldsymbol{\varphi}_f(t_k, \boldsymbol{\theta}_j) = \Bigg[\frac{-p}{A_j(p)}y(t_k), \ldots, \frac{-p^n}{A_j(p)}y(t_k), \left\{\frac{1}{A_j(p)}u(t)\right\}_{t=t_k}, \ldots, \left\{\frac{p^m}{A_j(p)}u(t)\right\}_{t=t_k}\Bigg]^\top. \quad (9)$$

Remark 4.1: In terms of asymptotic covariance, it can also be shown (see [9] for the details for the open-loop case) that the correct intersample behavior of the input (reference) in the filtered instrument is needed for asymptotic efficiency of the method. Thus, we need to compute

$$\hat{\boldsymbol{\varphi}}_f(t_k, \boldsymbol{\theta}_j) = \Bigg[\frac{-pB_j(p)}{A_j^2(p)}u(t), \ldots, \frac{-p^nB_j(p)}{A_j^2(p)}u(t), \frac{1}{A_j(p)}u(t), \ldots, \frac{p^m}{A_j(p)}u(t)\Bigg]^\top_{t=t_k} \quad (10)$$

for the open loop case, and, for the closed-loop scenario,

$$\hat{\boldsymbol{\varphi}}_f(t_k, \boldsymbol{\theta}_j) = \Bigg[\frac{-pT_{o,j}(p)}{A_j(p)}r(t), \ldots, \frac{-p^nT_{o,j}(p)}{A_j(p)}r(t), \frac{S_{uo,j}(p)}{A_j(p)}r(t), \ldots, \frac{p^mS_{uo,j}(p)}{A_j(p)}r(t)\Bigg]^\top_{t=t_k}. \quad (11)$$

The filtering can be done by solving the associated ordinary differential equation of each case via, e.g., Runge-Kutta methods. However, this may add extra computational burden to the procedures, as we are only interested in the filtered input and reference at the time instants $\{t_k\}_{k=1}^N$. By introducing a Delta transform description, the computations in (9), (10) and (11) can be generalized for an arbitrary input signal with arbitrary accuracy, thus reducing the interpolation errors when filtering the input. The proposed algorithm for computing $\boldsymbol{\varphi}_f(t_k, \boldsymbol{\theta}_j)$ and $\hat{\boldsymbol{\varphi}}_f(t_k, \boldsymbol{\theta}_j)$ at the j-th iteration of the SRIVC and CLSRIVC methods goes as follows:

1) Given the sampling period h of $y(t)$, (over) sample $u(t)$ (and possibly $r(t)$ as well) by a factor $S \gg 1$.
2) From $\boldsymbol{\theta}_j$, form a state-space realization of the prefilters of interest, namely $p^iB_j(p)/A_j^2(p)$ and $p^l/A_j(p)$ for $i = 1, \ldots, n$; $l = 0, \ldots, m$ for the open loop framework, and also $p^iT_{o,j}(p)/A_j(p)$ and $p^lS_{uo,j}(p)/A_j(p)$ for closed-loop.
3) Compute the Delta equivalent [18] of the prefilters in state-space form, i.e.,
$$\mathbf{x}(kh + h/S) = \mathbf{A}_\delta\mathbf{x}(kh) + \mathbf{B}_\delta\mathbf{w}(kh)$$
$$\mathbf{z}(kh) = \mathbf{C}_\delta\mathbf{x}(kh) + \mathbf{D}_\delta\mathbf{w}(kh),$$
where $\mathbf{z}(kh)$ denotes the output of the filtering process and $\mathbf{w}(kh) = u(kh)$ for open-loop, and $\mathbf{w}(kh) = [u(kh), r(kh)]^\top$ for the closed-loop case.
4) Calculate the response at instants $t_k = kh$ of each filter to the fast-sampled version of $u(t)$ in the Delta domain:
$$\mathbf{x}([k+1]h) = \mathbf{A}_\delta^S\mathbf{x}(kh) + \sum_{l=0}^{S-1}\mathbf{A}_\delta^{S-1-l}\mathbf{B}_\delta\mathbf{w}(h[k+l/S])$$
$$\mathbf{z}(kh) = \mathbf{C}_\delta\mathbf{x}(kh) + \mathbf{D}_\delta\mathbf{w}(kh).$$

The modified SRIVC and CLSRIVC methods with prefilters computed as in steps 1 to 4 above calculate the filtered regressor and instrument vectors accurately if the oversampling rate S is large, and exact simulation is achieved for $S \to \infty$. In practice, we have found that sampling 100 times faster is usually enough to provide reliable estimates. Due to potentially high sampling rates, the use of the Delta operator is needed for ameliorating rounding errors and ill-conditioning problems regarding the sensitivity of the coefficients of the prefilters.

A. Particular case: Band-limited signals

The method described in the previous section can be used in any situation when an arbitrary continuous-time input signal is recorded. Here we propose an alternative procedure to compute the filtering steps for when the input is a band-limited signal. For the following approach, we only require even samples of the input or reference signal, and oversampling is not required. Before we present the method, we briefly review concepts of band-limited signals.

A *band-limited* signal is a continuous-time signal that does not have energy above a certain frequency ω_B. That is, $U(i\omega) = 0$ for $|\omega| > \omega_B$, where

$$U(i\omega) = \int_{-\infty}^{\infty} u(t)e^{-i\omega t}\mathrm{d}t.$$

If $u(t)$ is sampled every h seconds, where $h < \pi/\omega_B$, then its discrete-time Fourier transform pair is given by

$$U_h(e^{i\omega h}) = h\sum_{k=-\infty}^{\infty} u(kh)e^{-i\omega kh} \iff u(kh) = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} U_h(e^{i\omega h})\frac{e^{i\omega kh}}{2\pi}\mathrm{d}\omega.$$

These expressions can be exploited so that the discrete-time Fourier transform is written in terms of the continuous-time one, which is known as Poisson's summation formula [19]

$$U_h(e^{i\omega h}) = \sum_{n=-\infty}^{\infty} U\left(i\omega + i\frac{2\pi n}{h}\right).$$

Due to $u(t)$ being band-limited, this formula indicates that $U_h(e^{i\omega h}) = U(i\omega)$ for $|\omega| < \pi/h$.

Our interest lies in computing the output of a single-input multi-output (SIMO) linear system such as (10), (11) or the last $m+1$ elements of (9), when the input $u(t)$ or reference $r(t)$ are band-limited. Without loss of generality, we consider an input $u(t)$ and a SIMO transfer function $\mathbf{H}(p)$. In the frequency domain, the continuous-time Fourier transform of the output is $\mathbf{Z}(i\omega) = \mathbf{H}(i\omega)U(i\omega)$. Therefore, the output $\mathbf{z}(t)$ is also band-limited and $\mathbf{Z}_h(e^{i\omega h}) = \mathbf{Z}(i\omega)$, also for $|\omega| < \pi/h$. Using these identities, we can reconstruct the output of an LTI system with band-limited excitation. By leveraging the inverse Fourier transform of $\mathbf{Z}_h(e^{i\omega h})$ we have

$$\mathbf{Z}(kh) = \frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \mathbf{H}(i\omega)U_h(e^{i\omega h})e^{i\omega kh}\mathrm{d}\omega$$
$$= \frac{1}{\pi}\mathrm{Re}\left\{\int_0^{\frac{\pi}{h}} \mathbf{H}(i\omega)U_h(e^{i\omega h})e^{i\omega kh}\mathrm{d}\omega\right\},$$

where we have used the fact that the Fourier transform of a real signal is conjugate symmetric. Thus, the output can be computed by solving N integrals. This is computationally expensive, considering that such integration must be performed at each iteration of the SRIVC algorithm and these integrals usually do not have a closed form. Instead of this, we evaluate its associated M+1-point Riemann sum

$$\mathbf{z}(kh) \approx \frac{1}{\pi h(M+1)}\mathrm{Re}\left\{\sum_{l=0}^{M} \mathbf{H}\left(\frac{i\pi l}{hM}\right)U_h(e^{i\frac{\pi l}{M}})e^{i\frac{\pi lk}{M}}\right\}.$$

Since the input samples $u(0), \dots, u([N-1]h)$ are known, we can compute $U_h(e^{i\frac{\pi l}{M}})$ as the zero-padded Discrete Fourier Transform (DFT) of the input vector. More precisely,

$$U_h(e^{i\frac{\pi l}{M}}) = h\sum_{k=0}^{N-1} u(kh)e^{-i\frac{\pi lk}{M}}.$$

Once $\{U_h(e^{i\frac{\pi l}{M}})\}_{l=0}^{M}$ is computed, the output vector $\{\mathbf{z}(kh)\}_{k=0}^{N-1}$ is obtained via the (zero-padded) inverse DFT of $\mathbf{H}\left(\frac{i\pi l}{hM}\right)U_h(e^{i\frac{\pi l}{M}})$.

Remark 4.2: For the computation of the filtering step described above, the choice of M dictates the accuracy of the filtering procedure: a larger value of M provides more accurate results at the expense of more arithmetic operations.

V. Simulation studies

We now study the performance of the SRIVC and CLSRIVC estimators under open and closed-loop examples and compare them with their proposed alternatives with regressor and instrument vectors computed via oversampling, which will be denoted as SRIVC-os and CLSRIVC-os in the sequel. The SRIVC and CLSRIVC estimators are computed from the CONTSID toolbox version 7.3 for MATLAB [20], under default initialization, tolerance $\epsilon = 10^{-10}$ and `MaxIter` $= 100$. The commands are set to estimate the best model within the correct model structure, and the monic denominator of the SRIVC estimator given by the CONTSID toolbox is converted to the form in (1) by dividing both polynomials by the constant term in the denominator.

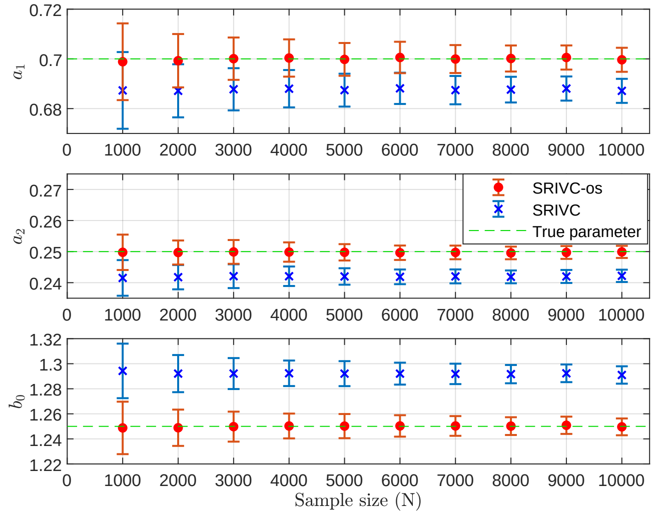

Fig. 2. Test with chirp input signal. Sample means of each parameter with 1 standard deviation for SRIVC-os (red circles) and standard SRIVC (blue crosses). The true parameter values are in dashed green.

A. Open-loop with a chirp signal input

Consider the second-order system

$$G^*(p) = \frac{1.25}{0.25p^2 + 0.7p + 1}, \tag{12}$$

where the parameters of interest are $a_1^* = 0.7$, $a_2^* = 0.25$ and $b_0^* = 1.25$. This system is excited with an up-chirp signal, which is a continuous-time signal that increases in frequency with time. These signals are widely used in signal processing applications such as radar systems and seismology and have been used for system identification [21], [22]. The chirp signal used in this example is

$$u(t) = \cos\left(f_0\left[\frac{f_1}{f_0}\right]^{t/T_f} 2\pi t\right),$$

where $[f_0, f_1] = [0.1, 0.6]$[Hz] and $T_f = 500$[s] is the length of one period of the chirp signal. We determine the true system output using the explicit Runge-Kutta formulae RK5(4). The output is sampled every $h = 0.5$[s] with $S = 500$, and the measurement noise has variance 0.05, which corresponds to approximately 10% of the energy of the noiseless output. For the computation of the SRIVC-os estimate, we follow the algorithm described in Section IV. The number of periods of the input varies from 1 to 10, which is equivalent to sample sizes ranging from 1000 to 10000, and 300 Monte Carlo runs are performed for each sample size.

The empirical evidence in Figure 2 suggests that the proposed procedure for computing the regressor and instrument vectors leads to accurate estimates of the parameters of interest, while the standard SRIVC method fails to deliver statistically consistent estimates. This improvement in accuracy is not without a drawback in computational efficiency: the total computation time of the SRIVC-os estimator tests is 30.2 [h], while the SRIVC estimates is computed in 1.3 [h]. The simulations are performed on a laptop with an Intel Core i7-7600u 2.8Ghz processor.

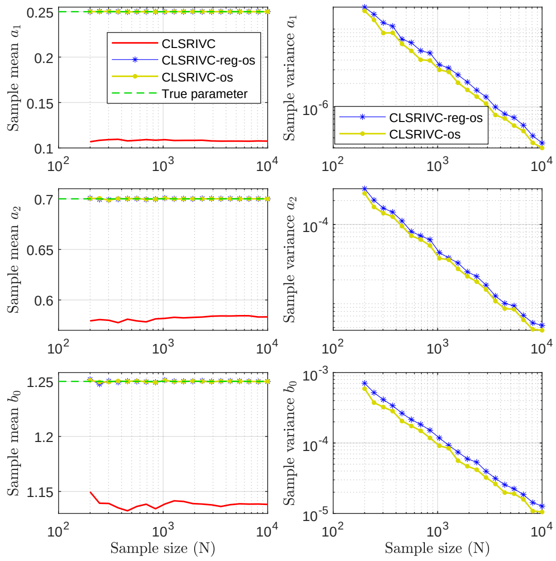

Fig. 3. Closed loop test. Left: Sample means of each parameter for CLSRIVC (straight, red), CLSRIVC-reg-os (blue asterisks), CLSRIVC-os (golden circles), and the true parameter (dashed green). Right: Sample variances of each parameter for CLSRIVC-reg-os and CLSRIVC-os.

B. Closed-loop with a band-limited reference signal

We consider that the system in (12) is now in closed-loop, where $C(p) = (p+3)/(5p)$, and $y(t)$ is sampled at the same rate as the previous example. The reference has been generated by a discrete-time Gaussian white noise of unit variance whose intersample behavior is given by a sinc interpolation, and is thus band-limited. The Gaussian white noise $v(t)$ is assumed to stay constant between samples[2] and has variance 0.01. The oversampling rate is set to $S = 100$. Twenty different sample sizes are considered, ranging logarithmically from $N = 200$ to $N = 10000$, and 300 Monte Carlo runs are performed for each value of N.

Figure 3 shows the sample means and sample variances of each estimated parameter, for three algorithms: standard CLSRIVC, CLSRIVC-os but with instrument vector computed with a ZOH intersample behavior in $r(t_k)$ as in (5) (labeled CLSRIVC-reg-os), and CLSRIVC-os (i.e., with regressor (9) and instrument (11)). Note that for CLSRIVC-os we compute the instrument vector following the procedure detailed in Section IV-A with $M = 100$. The results show that the oversampling procedure for computing the regressor can cancel the bias in each parameter, independent on how the instrument is formed. However, if the instrument is also computed via oversampling, the variance of each parameter is also reduced compared to the standard ZOH filtering that is performed for the CLSRIVC-reg-os estimator. These findings agree with the theoretical derivation in Section III and Remark 4.1.

VI. Conclusions

In this paper, we have studied the effect of having a misspecified intersample behavior of the input and reference signals in the SRIVC and CLSRIVC estimators in continuous-time system identification. These methods have been proven to be inconsistent if the intersample behavior of the signals are not taken into consideration. Our approach to solve this problem consists in computing the filtering steps of each method via over-sampling of arbitrary signals or computing the filtering step exactly if the signal is band-limited. Via numerical tests, we observe that our methods offer a promising way to alleviate the asymptotic bias.

[2]This hybrid formulation, consisting in the mixture between continuous-time and discrete-time signals, is commonly used in closed-loop continuous-time system identification; see e.g. [1, Chapter 5].

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