Consistency analysis and bias elimination of the Instrumental-Variable-based State Variable Filter method ⋆

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Abstract

In this paper, we analyse the consistency of the Instrumental-Variable-based State Variable Filter (IVSVF) estimator by taking into account the intersample behaviour of the input and output signals. It is found that when only sampled input and output data are available for estimation, the IVSVF estimator is not consistent for any fixed sampling period due to the interpolation error that arises from constructing the filtered output signal. A Bias-Eliminated IVSVF (BEIVSVF) estimator is then proposed and shown to be consistent. The theoretical results developed in the paper are also discussed from a practical standpoint. Simulations are performed to verify the performance of the proposed method as well as to support the theoretical results.

Key words: Consistency, CT system, Identification, Instrumental variable method

1 Introduction

Dynamical systems in the physical world are generally continuous-time (CT) in nature, hence it is more intuitive to describe these systems using CT models. CT system identification can be classified into the direct and the indirect approaches [1]. The indirect approach identifies a discrete-time (DT) model and then transforms it to CT, whereas the direct approach identifies a CT model directly from sampled data. There are several advantages associated with the direct approach [1], such as the direct link between model coefficients and physical parameters of the system and the relative ease of identifying models from irregularly sampled data. Nevertheless, there are also issues associated with the direct approach, such as the requirement for the time-derivatives of the input and output. Direct differentiation of these signals is not a viable option for obtaining time-derivatives as measured signals are usually contaminated with noise.

The use of prefilters [15,24] was suggested to avoid the problem of direct differentiation. This prefiltering method is now known as the State Variable Filter (SVF) approach [1]. The Least Squares (LS) method was then used in combination with the SVF (LSSVF) to estimate CT models [17]. After realising that the LSSVF estimator may be asymptotically biased, the Instrumental-Variable-based State Variable Filter (IVSVF) estimator was introduced to alleviate this bias [18,19]. Early research considered either a completely *analog implementation* (see e.g. [16]) or a hybrid *analog-digital implementation* (see e.g. [17,19]) of the algorithm, i.e. the filtered derivatives were obtained using analog filters whereas the estimation algorithm was implemented in a digital device.

Due to the advancement of digital computers, the use of analog devices in the CT estimation algorithm implementation became less popular, and DT estimation methods became more prevalent. For instance, an iterative Refined Instrumental Variable (RIV) method was proposed in the late 70's [20,22] for identifying DT systems. In addition, a CT version of this algorithm was developed [23], known as the RIVC estimator, for identifying hybrid Box-Jenkins models, i.e. where the system model is in CT and the noise model in DT. Furthermore, a simplified version of the RIVC estimator, known as the SRIVC estimator [23], was also introduced for identifying CT systems in an output error model structure. It should be noted that the "continuous-discrete" imple-

⋆ This paper was not presented at any IFAC meeting.
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mentation of the SRIVC and RIVC estimators [23] had assumed that only sampled input and output were available as measurements, i.e. the prefilters and the estimators were both implemented digitally. This has been the standard way of implementing direct CT estimator since the prevalence of digital computers. In the current paper, this approach is referred to as the *digital implementation.* It, however, raises an important implementation issue regarding the intersample behaviour of the signals.

It has been proven that the SRIVC estimator is consistent [8] under mild conditions if the intersample behaviour of the input in the regressor vector matches that of the system input. Provided that the interpolation error that arises from constructing the filtered output does not affect the non-singularity of the modified normal matrix, the intersample behaviour of the output neither impacts on the consistency nor the efficiency of the SRIVC estimator as shown in [8] and [10], respectively. This is due to the refinement of the prefilters in the SRIVC estimator, which, for finite sample sizes, allows the same estimate to be obtained at the converging point regardless of the intersample behaviour of the sampled output [10, Lemma 6]. This phenomenon does not occur with the IVSVF estimator since it uses a fixed choice of denominator in its prefilter. Consequently, the digital implementation of the CT filter for the sampled output will have an impact on the IVSVF estimates. This output interpolation problem has been overlooked in the literature (see e.g. [1, p. 263] and [6]) where it is claimed that the IVSVF estimator is asymptotically unbiased.

The objectives of this paper are twofold. Firstly, it is proven that the IVSVF estimator is not consistent in a digital implementation, i.e. when only sampled signals are available for estimation. This is due to the interpolation error that arises when filtering the output. The asymptotic bias of the IVSVF estimator, which has not been addressed adequately in the existing literature, is demonstrated in simulation studies and more importantly shown theoretically in the current paper. The implications and consequences of filtering signals in CT estimators are also highlighted. Secondly, a Bias-Eliminated IVSVF (BEIVSVF) estimator is proposed to eliminate the bias that arises in the digital implementation of the IVSVF estimator. The consistency of the proposed estimator is shown theoretically and its performance evaluated in simulation studies. The choice of the hyperparameter, specifying the bandwidth of the SVF, in the IVSVF and BEIVSVF estimators is then discussed thoroughly with respect to the theoretical results developed in the current paper. In addition, the practical implications of using the estimators can be more easily revealed from the established theorems and therefore are also discussed.

This paper is organised as follows. Section 2 highlights the differences between the classical analog-digital implementation and the modern digital implementation of the IVSVF estimator. This is followed by Section 3 where the consistency analysis of the IVSVF estimator is presented. The proposed BEIVSVF estimator and its consistency analysis are given in Section 4. Section 5 provides simulation results that support the theoretical analyses, and the paper is concluded in Section 6.

2 Preliminaries

In this section, the CT system and model definitions are provided with respect to two different implementations of the IVSVF estimator. The first one is a *classical analog-digital* implementation that employs analog filtering of the input and output signals followed by a digital estimation procedure. The second one is a *digital* implementation, where both the filtering of the signals and estimation procedure are performed digitally.

2.1 Classical analog-digital implementation of IVSVF

Consider a linear time-invariant CT system

$$\mathcal{S}\colon \begin{cases} x^*(t) &= \dfrac{B^*(p)}{A^*(p)}u(t) \\ y(t) &= x^*(t) + v(t), \end{cases} \tag{1}$$

where p is the differential operator, i.e. $py(t) = \frac{d}{dt}y(t)$, $x^*(t)$ the noise-free system output, $u(t)$ the system input and $v(t)$ the additive noise on the output. The system numerator and denominator polynomials are assumed coprime with orders given by m^* and n^* respectively, i.e.

$$\begin{aligned} B^*(p) &= b_0^* p^{m^*} + b_1^* p^{m^*-1} + \cdots + b_{m^*}^*, \\ A^*(p) &= a_1^* p^{n^*} + a_2^* p^{n^*-1} + \cdots + a_{n^*}^* p + 1, \end{aligned}$$

with the system parameter vector given by

$$\boldsymbol{\theta}^* := \begin{bmatrix} a_1^* & \dots & a_{n^*}^* & b_0^* & \dots & b_{m^*}^* \end{bmatrix}^\top .$$

A CT model is needed in order to derive the classical LSSVF estimator, i.e.

$$\mathcal{M}\colon y(t) = \frac{B(p)}{A(p)}u(t) + e(t) \tag{2}$$

with coprime numerator and denominator polynomials,

$$\begin{aligned} B(p) &= b_0 p^m + b_1 p^{m-1} + \cdots + b_m, \\ A(p) &= a_1 p^n + a_2 p^{n-1} + \cdots + a_n p + 1, \end{aligned} \tag{3}$$

where the model parameter vector is given by

$$\boldsymbol{\theta} := \begin{bmatrix} a_1 & \dots & a_n & b_0 & \dots & b_m \end{bmatrix}^\top ,$$

and $e(t)$ is the residual.

To highlight the difference between the analog and digital implementations, the following two notations are adopted.

Notation 1 *A CT signal, $x(t)$, is filtered prior to being sampled when the following filtering notation is used, i.e.*

$$\{Q(p)x(t)\}_{t=t_k},$$

where $Q(p)$ is an arbitrary CT transfer function. Square brackets are used when vectors are considered.

Notation 2 *A mixed notation of CT operators and DT data, i.e.*

$$Q(p)x(t_k),$$

implies that the sampled signal $x(t_k)$ is interpolated with some intersample behaviour assumption such as a zero-order hold (ZOH) or a first-order hold (FOH), and the filtered signal is then sampled at t_k.

Note that the filtering convention used in Notation 2 has been the standard way of expressing DT data being filtered by CT transfer functions in the existing literature (see e.g. [1]).

The model in (2) can be rewritten as

$$A(p)y(t) = B(p)u(t) + A(p)e(t). \tag{4}$$

To avoid direct differentiation of the signals in (4), the SVF approach [24,17] is considered, where a linear filter, i.e. the SVF with $\lambda > 0$, given by

$$\frac{1}{F(p)} = \frac{1}{(p/\lambda+1)^n} = \frac{1}{f_1p^n + f_2p^{n-1} + \cdots + 1}, \tag{5}$$

is applied to both sides of (4), that is

$$\frac{1}{F(p)}A(p)y(t) = \frac{1}{F(p)}B(p)u(t) + \frac{1}{F(p)}A(p)e(t). \tag{6}$$

The relationship in (6) is equivalent to that of (4) in terms of the unknown parameters, $\boldsymbol{\theta}$, subsequent to a time, t_0, provided that the initial condition of the SVF has negligible effects on the filtered input and output after t_0. Furthermore, the hyperparameter, λ, is required to be chosen such that the bandwidth of $1/F(p)$ contains the bandwidth of the system [1] as suggested in the existing literature (see e.g. [19]).

Since the system is assumed to be time-invariant, the linear filter $1/F(p)$ commutes with $A(p)$ and $B(p)$, thus (6) can be rewritten as

$$\begin{aligned} a_1\mathring{y}_f^{(n)}(t) + \cdots + a_n\mathring{y}_f^{(1)}(t) + \mathring{y}_f(t) \\ = b_0u_f^{(m)}(t) + \cdots + b_mu_f(t) + \eta(t), \end{aligned} \tag{7}$$

where $\mathring{y}_f^{(i)}(t) = \frac{p^i}{F(p)}y(t)$, $u_f^{(i)}(t) = \frac{p^i}{F(p)}u(t)$ and $\eta(t)$ is the filtered residual vector. The filtered outputs in (7) are accentuated with a circle, i.e. $\mathring{y}_f^{(i)}$, to distinguish the signals generated by an analog filter from those generated by a digital filter, i.e. $y_f^{(i)}$. This digitally filtered output will be encountered in Section 2.2. Note that these two signals are different even if the same transfer function is used since the intersample behaviour of the output is unknown in a digital implementation, thus the interpolation is never exact. Filtered input signals do not use this notation since the input intersample behaviour is assumed to be known exactly in the analysis [2].

Historically, an analog-digital implementation of this CT estimator was employed (see e.g. [19]), where the system input and output are filtered by analog SVF's to obtain the filtered derivatives, then the signals are sampled and the estimator is implemented in a digital device. Hence, after sampling the filtered signals regularly at $t = t_k$, (7) can be expressed in a linear regression form as

$$\mathring{y}_f(t_k) = \mathring{\boldsymbol{\varphi}}_f^\top(t_k)\boldsymbol{\theta} + \eta(t_k), \tag{8}$$

where the regressor vector and filtered output in sampled form are given by

$$\begin{aligned} \mathring{\boldsymbol{\varphi}}_f(t_k) = \Big[-\tfrac{p^n}{F(p)}y(t) \ \dots \ -\tfrac{p}{F(p)}y(t) \\ \tfrac{p^m}{F(p)}u(t) \ \dots \ \tfrac{1}{F(p)}u(t) \Big]^\top_{t=t_k}, \end{aligned} \tag{9}$$

and

$$\mathring{y}_f(t_k) = \left\{ \frac{1}{F(p)}y(t) \right\}_{t=t_k}, \tag{10}$$

respectively. The regressor vector in (9) is accentuated with a circle as it contains the filtered output generated using analog filters. Similarly, this notation will also be applied to the resulting estimators, $\mathring{\boldsymbol{\theta}}_{ls}$ and $\mathring{\boldsymbol{\theta}}_{iv}$, as shown in (11) and (12).

Now, from N samples of the input and output filtered derivatives, the LSSVF estimator is given by

$$\mathring{\boldsymbol{\theta}}_{ls} = \left[\frac{1}{N}\sum_{k=1}^{N} \mathring{\boldsymbol{\varphi}}_f(t_k)\mathring{\boldsymbol{\varphi}}_f^\top(t_k) \right]^{-1} \left[\frac{1}{N}\sum_{k=1}^{N} \mathring{\boldsymbol{\varphi}}_f(t_k)\mathring{y}_f(t_k) \right]. \tag{11}$$

[1] This point will be discussed in detail in Section 4.3 as it is a loose statement on the requirement of the bandwidth of the SVF.

[2] If the intersample behaviour of the input is not known exactly, then the estimator will be inconsistent, which can be inferred from the results in [8].

It is well known that, in CT system identification, the LS estimator in (11) is biased even if the equation error is i.i.d. white noise [1]. This asymptotic bias problem can be alleviated by the use of the IVSVF estimator [19] given by

$$\mathring{\boldsymbol{\theta}}_{iv}=\left[\frac{1}{N}\sum_{k=1}^{N}\hat{\boldsymbol{\varphi}}_f(t_k,\mathring{\boldsymbol{\theta}}_{ls})\mathring{\boldsymbol{\varphi}}_f^\top(t_k)\right]^{-1}\left[\frac{1}{N}\sum_{k=1}^{N}\hat{\boldsymbol{\varphi}}_f(t_k,\mathring{\boldsymbol{\theta}}_{ls})\mathring{y}_f(t_k)\right], \tag{12}$$

where $\hat{\boldsymbol{\varphi}}_f(t_k,\mathring{\boldsymbol{\theta}}_{ls})$ is known as the instrument vector, which is chosen to be highly correlated with the input but uncorrelated with the disturbance. One option to form the instrument vector is to replace the measured output in the regressor by the estimated model output, i.e.

$$\hat{\boldsymbol{\varphi}}_f(t_k,\mathring{\boldsymbol{\theta}}_{ls})=\Big[-\tfrac{p^n}{F(p)}x_{ls}(t)\ \dots\ -\tfrac{p}{F(p)}x_{ls}(t)\quad \tfrac{p^m}{F(p)}u(t)\ \dots\ \tfrac{1}{F(p)}u(t)\Big]^\top_{t=t_k}, \tag{13}$$

where $x_{ls}(t)$ is known as the auxiliary signal. It is generated using an auxiliary model with parameters given by the LSSVF estimator (11).

2.2 Digital implementation of IVSVF

It is important to reiterate that the LSSVF and IVSVF estimators discussed in Section 2.1 were originally derived for a CT model where the prefiltering was performed with analog filters. Since the prevalence of digital computers, both the prefiltering process and estimation algorithms are implemented digitally. In this section, we will make some modifications to the CT model description and the SVF methods to reflect the nature of the digital implementation of the estimators.

In a digital implementation of the LSSVF and IVSVF estimators, the intersample behaviour assumption of the signals when performing filtering options becomes important. We will assume that the intersample behaviour of the system input is known such that the filtered version of the input can be reconstructed exactly and only analyse the consistency of the IVSVF estimator with respect to the intersample behaviour of the output. This assumption is imposed as it has been shown [8] that the SRIVC estimator, which can be thought of as an iterative and refined version of the IVSVF estimator, is generically inconsistent when an incorrect input intersample behaviour is used.

Since only sampled input and output are available in practice, the output observation equation is expressed as

$$y(t_k)=\left\{\frac{B^*(p)}{A^*(p)}u(t)\right\}_{t=t_k}+v(t_k),$$

where the additive noise on the output, $v(t_k)$, is a zero-mean DT stochastic random process.

In the digital implementation of the direct CT estimators, it is more natural to adopt Notation 2 to represent filtered signals due to the nature of the sampled data. However, if the intersample behaviour of the input is known exactly, then Notations 1 and 2 are equivalent in terms of the filtered signals they represent. Notation 1 is chosen to represent the filtered input in the current paper in order to combine the two types of filtering required when representing the filtered system output in the consistency analysis. This point will be more thoroughly explained when encountered in Section 3.1.

For a digital implementation of the CT estimator, the model of the system is parameterised in (14) with a proper transfer function, i.e.

$$\mathcal{M}\colon\begin{cases} x(t_k) &= \left\{\dfrac{B(p)}{A(p)}u(t)\right\}_{t=t_k}\\ y(t_k) &= x(t_k)+e(t_k),\end{cases} \tag{14}$$

where $B(p)$ and $A(p)$ are given in (3). Since the SVF prefiltering is performed after the data has been sampled, the linear regression model in (8) is modified to be

$$y_f(t_k)=\boldsymbol{\varphi}_f^\top(t_k)\boldsymbol{\theta}+\varepsilon(t_k), \tag{15}$$

where the regressor is given by

$$\boldsymbol{\varphi}_f(t_k)=\Big[-\tfrac{p^n}{F(p)}y(t_k)\ \dots\ -\tfrac{p}{F(p)}y(t_k)\quad \left\{\tfrac{p^m}{F(p)}u(t)\right\}_{t=t_k}\ \dots\ \left\{\tfrac{1}{F(p)}u(t)\right\}_{t=t_k}\Big]^\top, \tag{16}$$

and the filtered output is

$$y_f(t_k)=\frac{1}{F(p)}y(t_k). \tag{17}$$

It is important to note the difference between the filtered outputs in (16) and (17) from those in (9) and (10). The residual term in (15) is known as the generalised equation error (GEE) of the model [19], which can be expressed as

$$\varepsilon(t_k)=\frac{A(p)}{F(p)}y(t_k)-\left\{\frac{B(p)}{F(p)}u(t)\right\}_{t=t_k}. \tag{18}$$

The LSSVF estimator minimises the sum of squares of the GEE of the model and its digital implementation version is obtained by modifying (11), i.e.

$$\boldsymbol{\theta}_{ls}=\left[\frac{1}{N}\sum_{k=1}^{N}\boldsymbol{\varphi}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right]^{-1}\left[\frac{1}{N}\sum_{k=1}^{N}\boldsymbol{\varphi}_f(t_k)y_f(t_k)\right].$$

The IVSVF estimator is employed to reduce the bias caused by the correlation between the LSSVF regressor vector and the residual. The digital implementation version of (12) is given by

$$\boldsymbol{\theta}_{iv}=\left[\frac{1}{N}\sum_{k=1}^{N}\hat{\boldsymbol{\varphi}}_f(t_k,\boldsymbol{\theta}_{ls})\boldsymbol{\varphi}_f^\top(t_k)\right]^{-1}\left[\frac{1}{N}\sum_{k=1}^{N}\hat{\boldsymbol{\varphi}}_f(t_k,\boldsymbol{\theta}_{ls})y_f(t_k)\right], \tag{19}$$

where the regressor vector $\boldsymbol{\varphi}_f(t_k)$ is given by (16), and the instrument vector is

$$\hat{\boldsymbol{\varphi}}_f(t_k,\boldsymbol{\theta}_{ls})=\Big[-\tfrac{p^nB_{ls}(p)}{F(p)A_{ls}(p)}u(t)\ \dots\ -\tfrac{pB_{ls}(p)}{F(p)A_{ls}(p)}u(t)\quad \tfrac{p^m}{F(p)}u(t)\ \dots\ \tfrac{1}{F(p)}u(t)\Big]^\top_{t=t_k}. \tag{20}$$

In the sequel, the argument $\boldsymbol{\theta}_{ls}$ will be omitted in the *digital* implementation version of the instrument vector for simplicity of notation, that is, (20) will be written as $\hat{\boldsymbol{\varphi}}_f(t_k)$. Note that the IVSVF estimator in (19) is implemented as a two-step method (i.e. LS followed by IV). An iterative version of this estimator [19] also exists and can be implemented with a fixed SVF by updating the auxiliary model in the instrument vector. However, as far as the consistency analysis is concerned in Section 3, the estimator will be considered non-iterative since it has been shown [8] that the instrument vector does not affect the consistency of the estimator as long as some mild conditions are satisfied, e.g. when chosen in the form of (20). Nevertheless, the covariance of the estimator can be affected [10].

Next, we provide a definition of generic non-singularity [12], which is closely related to the concept of generic consistency in Sections 3 and 4, and a lemma that examines this property.

Definition 1 *Consider an $n\times n$ matrix $\mathbf{R}(\boldsymbol{\rho})$, which depends on a vector $\boldsymbol{\rho}$ belonging to an open set Ω of the Euclidean space $\mathbb{R}^{n_\rho}$. Then, $\mathbf{R}$ is generically non-singular with respect to $\boldsymbol{\rho}\in\Omega$ if the set $\{\boldsymbol{\rho}:\boldsymbol{\rho}\in\Omega, \text{rank}\,\mathbf{R}(\boldsymbol{\rho})<n\}$ has Lebesgue measure zero in Ω.*

Lemma 1 *Consider the matrix $\mathbf{R}$ and the set Ω given in Definition 1. Assume that*

(i) The elements of $\mathbf{R}$ are analytic functions of every element of $\boldsymbol{\rho}\in\Omega$.
(ii) There is a vector $\boldsymbol{\rho}^*\in\Omega$ such that $\mathbf{R}(\boldsymbol{\rho}^*)$ is non-singular.

Then, $\mathbf{R}(\boldsymbol{\rho})$ is generically non-singular with respect to $\boldsymbol{\rho}\in\Omega$.

Proof of Lemma 1 By using Proposition 1 on the zero set of real analytic functions in [7], the proof of Lemma 1 then follows directly from the proof of Lemma 1 in [13].

3 Analysis of the IVSVF Estimator

This section provides the theoretical analysis showing that the digital implementation of the IVSVF estimator is not consistent. The implications of filtering sampled data in CT system identification are also discussed.

3.1 Consistency Analysis of the IVSVF Estimator

We begin the analysis by expressing the filtered regressor given in (16) as

$$\boldsymbol{\varphi}_f(t_k)=\tilde{\boldsymbol{\varphi}}_f(t_k)+\mathbf{v}_f(t_k)+\boldsymbol{\Delta}(t_k),$$

where $\mathbf{v}_f(t_k)$ contains the filtered version of the additive noise on the output, $\tilde{\boldsymbol{\varphi}}_f(t_k)$ is the noise-free version of $\mathring{\boldsymbol{\varphi}}_f(t_k)$ in (9), i.e.

$$\tilde{\boldsymbol{\varphi}}_f(t_k)=\Big[-\tfrac{p^nB^*(p)}{F(p)A^*(p)}u(t)\ \dots\ -\tfrac{pB^*(p)}{F(p)A^*(p)}u(t)\quad \tfrac{p^m}{F(p)}u(t)\ \dots\ \tfrac{1}{F(p)}u(t)\Big]^\top_{t=t_k},$$

and $\boldsymbol{\Delta}(t_k)$ is a vector that contains the interpolation error that arises from constructing the filtered output, which is given by the difference between the noise-free versions of $\boldsymbol{\varphi}_f(t_k)$ and $\mathring{\boldsymbol{\varphi}}_f(t_k)$, that is

$$\boldsymbol{\Delta}_i(t_k)=\begin{cases}\left\{\frac{p^{n+1-i}B^*(p)}{F(p)A^*(p)}u(t)\right\}_{t=t_k}-\frac{p^{n+1-i}}{F(p)}\left\{\frac{B^*(p)}{A^*(p)}u(t)\right\}_{t=t_k}, & i=1,\dots,n\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, we have

$$\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right\}=\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k)\tilde{\boldsymbol{\varphi}}_f^\top(t_k)\right\}+\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\Delta}^\top(t_k)\right\}+\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k)\mathbf{v}_f^\top(t_k)\right\}. \tag{21}$$

Note that we will refer to (21) as the modified normal matrix as IV methods replace the first regressor vector of the normal matrix in the LS method by the instrument vector.

Next, we state the assumptions and a lemma required in Theorems 1 and 2 regarding the consistency of the IVSVF and BEIVSVF estimators.

Assumption 1 *The system, $\frac{B^*(p)}{A^*(p)}$, is proper ($n^*\geq m^*$) and asymptotically stable with $A^*(p)$ and $B^*(p)$ being coprime.*

Assumption 2 *The input sequence, $u(t_k)$, and disturbance, $v(t_s)$, are stationary and mutually independent for all k and s.*

Assumption 3 *The input sequence, $u(t_k)$, is persistently exciting of order no less than $2n+1$.*

Assumption 4 *All the zeros of $A_{iv}(p)$ and $A_{ls}(p)$ have strictly negative real parts, $n \geq m$, with $A_{iv}(p)$ and $B_{iv}(p)$ being coprime as well as $A_{ls}(p)$ and $B_{ls}(p)$ being coprime.*

Assumption 5 *The degrees of the polynomials in the model satisfy* $\min(n-n^*, m-m^*) = 0$.

Assumption 6 *The sampling frequency is more than twice that of the largest imaginary part of the zeros of $F(p)A^*(p)$.*

Assumption 3 means that the support of the input spectral distribution function contains at least $2n+1$ points [5]. Unstable poles may arise when estimating the parameters of transfer functions. A simple way to deal with this is to reflect the unstable poles across the imaginary axis, which has been extensively used for direct CT estimators (see e.g. [1, Chapter 4]). A more sophisticated way of dealing with the instability issue is to constrain the estimated parameters in a convex stability domain as shown in [3]. Hence, Assumption 4 is commonly satisfied in practice.

Lemma 2 *Under Assumptions 1 – 6, the modified normal matrix* (21) *of the IVSVF estimator is generically non-singular with respect to the system and prefilters if*

$$\left\|\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\Delta}^\top(t_k)\right\}\right\|_2 < \sigma_{\min}\left(\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k)\tilde{\boldsymbol{\varphi}}_f^\top(t_k)\right\}\right), \tag{22}$$

where $\|\cdot\|_2$ denotes the induced 2-norm of the matrix and $\sigma_{\min}(\cdot)$ is the smallest singular value of the matrix.

Proof of Lemma 2 The generic non-singularity of (21) can be proven using the same procedure as in [9].

Lemma 2 ensures the existence of a solution for the IVSVF estimator. This can be achieved without having an exact interpolation of the filtered output. The condition in (22) is a requirement on the size of the interpolation errors that arise from constructing the filtered derivatives of the output, which depend on both the sampling period and the interpolation method. Linear interpolation or spline methods are usually recommended for constructing the filtered output depending on the nature of the output signal. In a practical situation, the non-singularity of (21) is often preserved for low order systems without the need for fast sampling. For high order systems, fast sampling rates should be considered due to the construction of higher order derivatives, which require more samples per cycle of the highest input frequency to represent the filtered derivatives adequately.

Next, we present a theorem that examines the consistency property of the IVSVF estimator.

Theorem 1 *Consider the IVSVF estimator defined in* (19)*, and suppose Assumptions 1 – 6 and condition* (22) *hold. Then, for a ZOH or FOH system input that can be reconstructed exactly, the IVSVF estimator is generically inconsistent with respect to the system and prefilters for any fixed sampling period when the algorithm is implemented based only on sampled input/output data.*

Proof of Theorem 1 Since $\boldsymbol{\varphi}_f(t_k)$ and $\hat{\boldsymbol{\varphi}}_f(t_k)$ in the modified normal matrix (19) are jointly stationary stochastic processes, according to [11, Lemma 3.1],

$$\frac{1}{N}\sum_{k=1}^{N}\hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k) \to \mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right\} \tag{23}$$

with probability 1. According to Lemma 2, (23) is generically non-singular. Together with Assumption 5, this implies that there exists a unique solution to (19) asymptotically.

Now, the IVSVF estimator, $\boldsymbol{\theta}_{iv}$, satisfies (19), which means that for large sample size, we have

$$\frac{1}{N}\sum_{k=1}^{N}\hat{\boldsymbol{\varphi}}_f(t_k)\left(y_f(t_k) - \boldsymbol{\varphi}_f^\top(t_k)\,\boldsymbol{\theta}_{iv}\right) = \mathbf{0}$$

$$\frac{1}{N}\sum_{k=1}^{N}\hat{\boldsymbol{\varphi}}_f(t_k)\varepsilon(t_k, \boldsymbol{\theta}_{iv}) = \mathbf{0}. \tag{24}$$

The GEE in (24) can be written as

$$\varepsilon(t_k, \boldsymbol{\theta}_{iv}) = \frac{A_{iv}(p)}{F(p)}\left\{\frac{B^*(p)}{A^*(p)}u(t)\right\}_{t=t_k} - \left\{\frac{B_{iv}(p)}{F(p)}u(t)\right\}_{t=t_k} + \frac{A_{iv}(p)}{F(p)}v(t_k). \tag{25}$$

As the sample size, N, approaches infinity, by [11, Lemma 3.1], (24) can be expressed as

$$\mathbb{E}\{\hat{\boldsymbol{\varphi}}_f(t_k)\varepsilon(t_k, \boldsymbol{\theta}_{iv})\} = \mathbf{0}. \tag{26}$$

In order to proceed, the transfer functions associated with the input in (25) are required to be combined together. It is important to note that the first term on the right-hand side of (25) uses both Notations 1 and 2. The difficulty is that, in the digital implementation of the algorithm, $B^*(p)/A^*(p)$ and $B_{iv}(p)/A_{iv}(p)$ are discretised according to the intersample behaviour of the input, whereas $A_{iv}(p)/F(p)$ is discretised according to the assumed intersample behaviour of the output. Hence, only the DT equivalents of these transfer functions can be combined despite the fact that they are expressed as CT transfer functions. To overcome this difficulty, recall from Section 2.1 that the output filtered by an analog SVF is given by

$$\mathring{y}_f(t_k) = \left\{ \frac{1}{F(p)} \frac{B^*(p)}{A^*(p)} u(t) \right\}_{t=t_k} + \left\{ \frac{1}{F(p)} v(t) \right\}_{t=t_k}. \tag{27}$$

Define $\varepsilon_u^*(t_k, \boldsymbol{\theta}_{iv})$ to be the reconstruction error of the noise-free filtered output, which can be viewed as a function of the system input, i.e.

$$\varepsilon_u^*(t_k, \boldsymbol{\theta}_{iv}) := \frac{A_{iv}(p)}{F(p)} \left\{ \frac{B^*(p)}{A^*(p)} u(t) \right\}_{t=t_k} - \left\{ \frac{A_{iv}(p)}{F(p)} \frac{B^*(p)}{A^*(p)} u(t) \right\}_{t=t_k}. \tag{28}$$

Now, we can express the GEE in (25) as

$$\begin{aligned} \varepsilon(t_k, \boldsymbol{\theta}_{iv}) &= \left\{ \frac{A_{iv}(p)}{F(p)} \frac{B^*(p)}{A^*(p)} u(t) \right\}_{t=t_k} - \left\{ \frac{B_{iv}(p)}{F(p)} u(t) \right\}_{t=t_k} \\ &\quad + \varepsilon_u^*(t_k, \boldsymbol{\theta}_{iv}) + \frac{A_{iv}(p)}{F(p)} v(t_k) \\ &= \left\{ \frac{A_{iv}(p)B^*(p) - B_{iv}(p)A^*(p)}{A^*(p)F(p)} u(t) \right\}_{t=t_k} \\ &\quad + \varepsilon_u^*(t_k, \boldsymbol{\theta}_{iv}) + \frac{A_{iv}(p)}{F(p)} v(t_k). \end{aligned} \tag{29}$$

Let

$$\begin{aligned} A_{iv}(p)B^*(p) - B_{iv}(p)A^*(p) &= h_0 p^r + h_1 p^{r-1} + \cdots + h_r \\ &:= H(p), \end{aligned} \tag{30}$$

where $r = \max(n + m^*, n^* + m) = n + m$. The GEE in (29) is then expressed as

$$\varepsilon(t_k, \boldsymbol{\theta}_{iv}) = \left\{ \frac{\mathbf{u}_d^\top(t)}{A^*(p)F(p)} \right\}_{t=t_k} \mathbf{h} + \varepsilon_u^*(t_k, \boldsymbol{\theta}_{iv}) + \frac{A_{iv}(p)}{F(p)} v(t_k), \tag{31}$$

where

$$\mathbf{u}_d(t) := \begin{bmatrix} u^{(n+m)}(t) & u^{(n+m-1)}(t) & \dots & u(t) \end{bmatrix}^\top, \tag{32}$$

and

$$\mathbf{h} = \begin{bmatrix} h_0 & h_1 & \dots & h_{n+m} \end{bmatrix}^\top.$$

The filtered instrument vector can be written as

$$\hat{\boldsymbol{\varphi}}_f(t_k) = \mathbf{S}(-B_{ls}, A_{ls}) \left\{ \frac{\mathbf{u}_d(t)}{A_{ls}(p)F(p)} \right\}_{t=t_k}, \tag{33}$$

where $\mathbf{S}(-B_{ls}, A_{ls})$ is a $(n+m+1) \times (n+m+1)$ Sylvester matrix formed using the coefficients of $B_{ls}(p)$ and $A_{ls}(p)$ (see e.g. [8] for the structure of the Sylvester matrix). This matrix is non-singular under the coprime condition in Assumption 4 according to [12, Lemma A3.2]. Now, substituting (33) and (31) into (26), we obtain

$$\begin{aligned} \mathbb{E}\left\{ \hat{\boldsymbol{\varphi}}_f(t_k) \varepsilon(t_k, \boldsymbol{\theta}_{iv}) \right\} &= \mathbf{S}(-B_{ls}, A_{ls}) \left(\boldsymbol{\Phi} \mathbf{h} + \boldsymbol{\Psi}_u + \boldsymbol{\Psi}_v \right) \\ &= \mathbf{0}, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\Phi} &= \mathbb{E}\left\{ \left\{ \frac{\mathbf{u}_d(t)}{A_{ls}(p)F(p)} \right\}_{t=t_k} \left\{ \frac{\mathbf{u}_d^\top(t)}{A^*(p)F(p)} \right\}_{t=t_k} \right\}, \\ \boldsymbol{\Psi}_u &= \mathbb{E}\left\{ \left\{ \frac{\mathbf{u}_d(t)}{A_{ls}(p)F(p)} \right\}_{t=t_k} \varepsilon_u^*(t_k, \boldsymbol{\theta}_{iv}) \right\}, \\ \boldsymbol{\Psi}_v &= \mathbb{E}\left\{ \left\{ \frac{\mathbf{u}_d(t)}{A_{ls}(p)F(p)} \right\}_{t=t_k} \frac{A_{iv}(p)}{F(p)} v(t_k) \right\}. \end{aligned}$$

Under Assumption 2, $\boldsymbol{\Psi}_v = \mathbf{0}$. It can be shown in a similar way as in [8, Theorem 1] that $\boldsymbol{\Phi}$ is generically non-singular. Together with the non-singularity of $\mathbf{S}(-B_{ls}, A_{ls})$, this then allows the coefficients of $H(p)$ to be represented by

$$\mathbf{h} = -\boldsymbol{\Phi}^{-1} \boldsymbol{\Psi}_u.$$

Note that $\boldsymbol{\Psi}_u \neq \mathbf{0}$ since the expression $\varepsilon_u^*(t_k, \boldsymbol{\theta}_{iv})$ given by (28) is input-dependent and generally not equal to zero in a digital implementation of the algorithm as a CT output cannot be reconstructed exactly. Hence, $\mathbf{h} \neq \mathbf{0}$ and from (30), we obtain

$$\frac{B_{iv}(p)}{A_{iv}(p)} = \frac{B^*(p)}{A^*(p)} - \frac{H(p)}{A_{iv}(p)A^*(p)},$$

i.e. the IVSVF solution does not correspond to the true system parameters asymptotically. Therefore, the IVSVF estimator (19) is not consistent. □

3.2 *Filtering Sampled Data in Direct CT System Identification Methods*

In practical situations, an important implication of Theorem 1 is that prefiltering should be performed with caution when using direct CT identification methods with sampled data. For instance, some methods may require the sampled input and/or output signals to be filtered prior to estimation, such as in the case of estimating a time delay (see e.g. [4]). It should be noted that if these prefilterings are performed prior to the estimation procedure, then the resultant estimate will most likely be biased since the intersample behaviours of the filtered signals are no longer known exactly. Thus, in situations where data prefiltering is required, it is recommended to combine any extra filtering of the sampled data with the existing prefiltering procedures by modifying the SRIVC/RIVC/IVSVF algorithms, i.e. the extra filters should be combined with the CT filters in the regressor and instrument vectors of the algorithms prior to discretisation. This means that for the SRIVC and RIVC estimators, the consistency property can be preserved since the intersample behaviour assumption of the input signal is not violated [8]. For the same reason, the existing bias on the IVSVF estimate, due to the output interpolation error, would not increase as compared to the estimates computed without the extra filtering steps.

4 The Bias-Eliminated IVSVF Estimator

A Bias-Eliminated IVSVF (BEIVSVF) estimator is proposed in this section to remove the asymptotic bias of the IVSVF estimator due to the interpolation error that arises from constructing the filtered output. The consistency property of the proposed BEIVSVF estimator is then proven in Section 4.2. The algorithm is of an iterative nature and is more computationally efficient than the well known SRIVC algorithm as it does not require the inverse of the modified normal matrix to be updated at every iteration. This computation efficiency, however, is not obtained without trading off some other properties of the estimator. These trade-offs are discussed at the end of this section.

4.1 The BEIVSVF Algorithm

It has been shown in the proof of Theorem 1 that the bias of the IVSVF estimator is caused by the output interpolation error, $\varepsilon_u^*(t_k, \boldsymbol{\theta}_{iv})$, in (28) being correlated with the input in the instrument vector. The idea of the BEIVSVF estimator is to approximate this interpolation error, estimate its corresponding parameter bias and then correct the current estimate with this bias. This can be done in an iterative manner, where the $(j+1)$-th iteration of the BEIVSVF estimator is given by

$$\boldsymbol{\theta}_{j+1} = \boldsymbol{\theta}_j + \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\varepsilon_u(t_k, \boldsymbol{\theta}_j)\right] \tag{34}$$

with $\hat{\boldsymbol{\varphi}}_f(t_k)$ and $\boldsymbol{\varphi}_f(t_k)$ given by (20) and (16) respectively, and

$$\varepsilon_u(t_k, \boldsymbol{\theta}_j) = \frac{A_j(p)}{F(p)}\left(y(t_k) - \left\{\frac{B_j(p)}{A_j(p)}u(t)\right\}_{t=t_k}\right).$$

Note that the initial estimate for the BEIVSVF estimator, $\boldsymbol{\theta}_1$, is obtained using the IVSVF estimator (19).

The algorithm is stopped when either a maximum number of iterations is reached or a measure of the relative error is smaller than a specified tolerance, i.e.

$$\frac{\|\boldsymbol{\theta}_{j+1} - \boldsymbol{\theta}_j\|_2}{\|\boldsymbol{\theta}_j\|_2} < \epsilon. \tag{35}$$

4.2 Consistency Analysis of the BEIVSVF Estimator

In this section, the consistency property of the BEIVSVF estimator is established in Theorem 2. In the proof of Theorem 2, we extend the denominator polynomial of the SVF, $F(p)$, to be any stable polynomial with order n. This assumption is made without any loss of generality since the denominator polynomial in (5) is considered as a particular choice of the generalised $F(p)$.

Theorem 2 (Generic consistency of the BEIVSVF estimator) *Consider the BEIVSVF estimator in* (34) *with the denominator of the SVF being an arbitrary stable n-th order polynomial. Suppose Assumptions 1 – 6 hold. Then, for a ZOH or FOH input, the following statements are true:*

(1) The modified normal matrix, $\mathbb{E}\{\hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\}$, is generically non-singular with respect to the system and prefilters, provided that the condition in (22) *is satisfied.*
(2) If the BEIVSVF estimator converges as the iterations, j, tends to infinity for sufficiently large sample size, N, then the true parameter, $\boldsymbol{\theta}^*$, is generically (with respect to the system and prefilters) the unique converging point.
(3) As the sample size, N, approaches infinity, the algorithm locally converges to $\boldsymbol{\theta}^*$ provided that the matrix $\mathbf{I} - \mathbf{Q}^{-1}\mathbf{R}$ has all eigenvalues with magnitude less than 1, where $\mathbf{I}$ is the identity matrix with dimension $n+m+1$,

$$\mathbf{Q} = \mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right\}, \tag{36}$$

and

$$\mathbf{R} = \mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k)\frac{\bar{A}(p)}{F(p)}\left\{\left[-\frac{p^n\bar{B}(p)}{\bar{A}^2(p)} \;\cdots\; -\frac{p\bar{B}(p)}{\bar{A}^2(p)} \;\; \frac{p^m}{\bar{A}(p)} \;\cdots\; \frac{1}{\bar{A}(p)}\right]u(t)\right\}_{t=t_k}\right\}. \tag{37}$$

Proof of Theorem 2, Statement 1 Since the modified normal matrix does not get updated in the BEIVSVF estimator, its generic non-singularity follows from Lemma 2. □

Proof of Theorem 2, Statement 2 Let the converging point of the BEIVSVF estimator be defined as

$$\bar{\boldsymbol{\theta}} := \lim_{j\to\infty} \boldsymbol{\theta}_j,$$

and let the corresponding model polynomials at the converging point be denoted as $\bar{A}(p)$ and $\bar{B}(p)$. Then, it can be seen from (34) that the BEIVSVF estimator at $\bar{\boldsymbol{\theta}}$ satisfies

$$\begin{aligned}\bar{\boldsymbol{\theta}} &= \bar{\boldsymbol{\theta}} + \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\varepsilon_u(t_k,\bar{\boldsymbol{\theta}})\right] \\ &= \bar{\boldsymbol{\theta}} + \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k) \right. \\ &\quad \left. \times \frac{\bar{A}(p)}{F(p)}\left(\left\{\frac{B^*(p)}{A^*(p)}u(t)\right\}_{t=t_k} + v(t_k) - \left\{\frac{\bar{B}(p)}{\bar{A}(p)}u(t)\right\}_{t=t_k}\right)\right].\end{aligned} \tag{38}$$

As $N \to \infty$, the sums in (38) can be replaced by their corresponding expected values [11, Lemma 3.1]. Since $\mathbb{E}\{\hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\}$ is non-singular according to Statement 1, (38) can be rearranged to be

$$\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k)\frac{\bar{A}(p)}{F(p)}\left\{\frac{B^*(p)}{A^*(p)}u(t) - \frac{\bar{B}(p)}{\bar{A}(p)}u(t)\right\}_{t=t_k}\right\} + \mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k)\frac{\bar{A}(p)}{F(p)}v(t_k)\right\} = \mathbf{0}. \tag{39}$$

By following the same procedure as [8, Theorem 1, Statement 2], it can be shown that

$$\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k)\frac{\bar{A}(p)}{F(p)}v(t_k)\right\} = \mathbf{0}$$

under Assumption 2. Hence, (39) simplifies to

$$\mathbb{E}\left\{\hat{\boldsymbol{\varphi}}_f(t_k)\frac{\bar{A}(p)}{F(p)}\left\{\frac{B^*(p)}{A^*(p)}u(t) - \frac{\bar{B}(p)}{\bar{A}(p)}u(t)\right\}_{t=t_k}\right\} = \mathbf{0},$$

which can also be written as

$$\mathbf{S}(-B_{ls}, A_{ls})\bar{\boldsymbol{\Phi}}\mathbf{h} = \mathbf{0},$$

where $\mathbf{S}(-B_{ls}, A_{ls})$ is the Sylvester matrix constructed using the parameters of the LSSVF estimator, $\mathbf{u}_d(t)$ is given by (32), $\mathbf{h}$ is the $(n+m+1)\times 1$ vector containing the coefficients of the polynomial $H(p) = A^*(p)\bar{B}(p) - B^*(p)\bar{A}(p)$ in descending order of degree, and

$$\bar{\boldsymbol{\Phi}} := \mathbb{E}\left\{\left\{\frac{\mathbf{u}_d(t)}{A_{ls}(p)F(p)}\right\}_{t=t_k} \frac{\bar{A}(p)}{F(p)} \left\{\frac{\mathbf{u}_d^\top(t)}{A^*(p)\bar{A}(p)}\right\}_{t=t_k}\right\}. \tag{40}$$

Now, consider the set of parameters that describe $A_{ls}(p)$ and $F(p)$, i.e.

$$\begin{aligned}\Omega = \{(a_1^{ls}, \ldots, a_n^{ls}, f_1, \ldots, f_n) \in \mathbb{R}^{2n} :\\ A_{ls}(p), F(p) \text{ are stable polynomials}\}.\end{aligned}$$

Define $\bar{\boldsymbol{\Phi}}^*$ as the matrix $\bar{\boldsymbol{\Phi}}$ in (40) with $A_{ls}(p) = A^*(p)$ and $\bar{A}(p) = F(p)$, that is

$$\bar{\boldsymbol{\Phi}}^* := \mathbb{E}\left\{\left\{\frac{\mathbf{u}_d(t)}{A^*(p)F(p)}\right\}_{t=t_k} \left\{\frac{\mathbf{u}_d^\top(t)}{A^*(p)F(p)}\right\}_{t=t_k}\right\}.$$

Then, $\bar{\boldsymbol{\Phi}}^*$ can be shown to be positive definite by [8, Lemma 7] for a FOH input signal with persistent excitation order no less than $2n+1$. The persistent excitation order can be relaxed to $2n$ for a ZOH input signal if the model is strictly proper. By following the same procedure as [8, Lemma 9], we can show that, for a fixed input and $\bar{A}(p)$, $\bar{\boldsymbol{\Phi}}$ is an analytic function of the parameters in Ω. Together with $\bar{\boldsymbol{\Phi}}^*$ being positive definite, we can conclude that $\bar{\boldsymbol{\Phi}}$ is generically non-singular with respect to Ω by Lemma 1. Hence, (40) implies $\mathbf{h} = \mathbf{0}$, which in turn means that

$$\frac{\bar{B}(p)}{\bar{A}(p)} = \frac{B^*(p)}{A^*(p)},$$

i.e. the converging point is generically unique and corresponds to the true parameters. ☐

Proof of Theorem 2, Statement 3 Now, we examine how the BEIVSVF estimate behaves around the converging point. The BEIVSVF estimate at the $(j+1)$-th iteration is given by

$$\boldsymbol{\theta}_{j+1} = \boldsymbol{\theta}_j + \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\varepsilon_u(t_k,\boldsymbol{\theta}_j)\right],$$

where

$$\varepsilon_u(t_k,\boldsymbol{\theta}_j) = \frac{A_j(p)}{F(p)}\left(y(t_k) - \left\{\frac{B_j(p)}{A_j(p)}u(t)\right\}_{t=t_k}\right).$$

Linearising $\boldsymbol{\theta}_{j+1}$ around $\bar{\boldsymbol{\theta}}$ gives

$$\boldsymbol{\theta}_{j+1} = \bar{\boldsymbol{\theta}} + \left.\frac{\partial \boldsymbol{\theta}_j}{\partial \boldsymbol{\theta}_j}\right|_{\boldsymbol{\theta}_j=\bar{\boldsymbol{\theta}}} (\boldsymbol{\theta}_j - \bar{\boldsymbol{\theta}}) + \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\varepsilon_u(t_k,\bar{\boldsymbol{\theta}})\right] + \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right]^{-1} \sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k) \left.\frac{\partial \varepsilon_u(t_k,\boldsymbol{\theta}_j)}{\partial \boldsymbol{\theta}_j}\right|_{\boldsymbol{\theta}_j=\bar{\boldsymbol{\theta}}} (\boldsymbol{\theta}_j - \bar{\boldsymbol{\theta}}) + o_p(\|\boldsymbol{\theta}_j - \bar{\boldsymbol{\theta}}\|),$$

which we then expand to obtain

$$\begin{aligned}
\boldsymbol{\theta}_{j+1} &= \bar{\boldsymbol{\theta}} + (\boldsymbol{\theta}_j - \bar{\boldsymbol{\theta}}) \\
&+ \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\varepsilon_u(t_k,\bar{\boldsymbol{\theta}})\right] \qquad \text{(41a)} \\
&+ \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k) \left.\frac{\partial A_j(p)}{\partial \boldsymbol{\theta}_j}\right|_{\boldsymbol{\theta}_j=\bar{\boldsymbol{\theta}}} \right. \\
&\left. \times \frac{1}{F(p)} \left(y(t_k) - \left\{ \frac{\bar{B}(p)}{\bar{A}(p)} u(t) \right\}_{t=t_k} \right) \right] (\boldsymbol{\theta}_j - \bar{\boldsymbol{\theta}}) \qquad \text{(41b)} \\
&- \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k)\boldsymbol{\varphi}_f^\top(t_k)\right]^{-1} \left[\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k) \frac{\bar{A}(p)}{F(p)} \right. \\
&\left. \times \left\{ \frac{\partial}{\partial \boldsymbol{\theta}_j} \left. \left(\frac{B_j(p)}{A_j(p)} \right) \right|_{\boldsymbol{\theta}_j=\bar{\boldsymbol{\theta}}} u(t) \right\}_{t=t_k} \right] (\boldsymbol{\theta}_j - \bar{\boldsymbol{\theta}}) \qquad \text{(41c)} \\
&+ o_p(\|\boldsymbol{\theta}_j - \bar{\boldsymbol{\theta}}\|), \qquad \text{(41d)}
\end{aligned}$$

where $o_p(\cdot)$ is the small-o notation that indicates convergence to zero in probability and is used to capture the linearisation error.

As $N \to \infty$, the sums in (41) can be replaced by their corresponding expectations [11, Lemma 3.1]. Then, (41a) goes to zero since the expression is evaluated at the converging point. By following the same procedure as [8, Theorem 1, Statement 3], (41b) can be shown to be zero asymptotically since the instrument vector is uncorrelated with some filtered version of the additive noise $v(t_k)$. Differentiating the transfer function in (41c) with respect to $\boldsymbol{\theta}_j$, we obtain

$$\frac{\partial}{\partial \boldsymbol{\theta}_j} \left. \left(\frac{B_j(p)}{A_j(p)} \right) \right|_{\boldsymbol{\theta}_j=\bar{\boldsymbol{\theta}}} = \frac{1}{\bar{A}(p)} \left[-p^n \tfrac{\bar{B}(p)}{\bar{A}(p)} \ \cdots \ -p \tfrac{\bar{B}(p)}{\bar{A}(p)} \ p^m \ \cdots \ 1 \right].$$

Hence, the second sum in (41c) becomes $\mathbf{R}$ in (37). Now, (41) simplifies to

$$\boldsymbol{\theta}_{j+1} - \bar{\boldsymbol{\theta}} = (\mathbf{I} - \mathbf{Q}^{-1}\mathbf{R})(\boldsymbol{\theta}_j - \bar{\boldsymbol{\theta}}) + o_p(\|\boldsymbol{\theta}_j - \bar{\boldsymbol{\theta}}\|), \qquad (42)$$

where $\mathbf{Q}$ and $\mathbf{R}$ are given by (36) and (37) respectively. Provided that all the eigenvalues of $\mathbf{I} - \mathbf{Q}^{-1}\mathbf{R}$ have magnitude less than 1, $\boldsymbol{\theta}_{j+1}$ is locally convergent to $\boldsymbol{\theta}^*$. □

The BEIVSVF estimator can also be shown to be generically consistent for a CT multisine input, which is stated in the following remark.

Remark 3 *The results in Theorem 2 still hold for a CT multisine input. Statement 1 and the generic non-singularity of (40) in Statement 2 can be shown by following the same procedure as in [2, Theorem 4].*

An important point to be noted here is that the interpolation error that arises from constructing the filtered output might affect the existence of a unique solution of the BEIVSVF estimator according to Lemma 2, however, it does not affect the converging point of the estimator. This means that, provided that the unique solution exists and the algorithm converges, which is usually attainable with a reasonable choice of the sampling period, the converging point will correspond to the true parameter in a digital implementation of the algorithm with ZOH, FOH or CT multisine inputs. This is contrary to the intuitive belief that direct CT methods are only consistent as the sampling period tends to zero. The above discussion also applies to the SRIVC estimator [8], although the SRIVC estimator has a stronger convergence result as explained in Section 4.4.

4.3 On the Choice of λ for the SVF

In Statement 2 of Theorem 2, we have proven that $\bar{\boldsymbol{\Phi}}$ in (40) is generically non-singular with respect to the parameters of $A_{ls}(p)$ and $F(p)$ in order to show that the unique converging point of the BEIVSVF estimator corresponds to the true parameter. According to Definition 1, the generic non-singularity of $\bar{\boldsymbol{\Phi}}$ implies that $\bar{\boldsymbol{\Phi}}$ is non-singular, and thus $\bar{\boldsymbol{\theta}} = \boldsymbol{\theta}^*$, for all selections of $F(p)$ except for some isolated cases. This means that, in a practical situation, $F(p)$ can be chosen to be almost any stable polynomial with the correct order without affecting the converging point of the estimator for a large sample size. Defining $F(p)$ to be in the form of $(p/\lambda+1)^n$ in (5) is considered as one of the possible parameterisations in the parameter space.

At the converging point, the BEIVSVF estimator solves the equation

$$\sum_{k=1}^{N} \hat{\boldsymbol{\varphi}}_f(t_k) \frac{\bar{A}(p)}{F(p)} \left(y(t_k) - \left\{ \frac{\bar{B}(p)}{\bar{A}(p)} u(t) \right\}_{t=t_k} \right) = \mathbf{0}. \qquad (43)$$

The expression in (43) gives some insights into the choice of $F(p)$. On the one hand, when the bandwidth of $1/F(p)$ is chosen to be much smaller than that of $1/\bar{A}(p)$, the information content that the input and output contain about the system will be greatly reduced, hence leading to poor estimates. On the other hand, if the bandwidth of $1/F(p)$ is chosen to be much larger than that of $1/\bar{A}(p)$, the high frequency region of this filter will have a much larger gain. This will in turn place more emphasis on the model fit in the high frequency region since the residual term in (43) is filtered by $\bar{A}(p)/F(p)$. Thus, the bandwidth of $1/F(p)$ should be chosen to be as close to $1/\bar{A}(p)$ as possible (note that, asymptotically, $\bar{A}(p) = A^*(p)$ since the estimator is consistent). Even though the choice of $F(p)$ does not affect the consistency property of the estimator, the quality of the estimate can deteriorate in a practical situation since only finite samples are considered, especially when the signal-to-noise ratio is low. Hence, from the above discussion, the user should exercise caution when choosing the hyperparameter λ for constructing the prefilter $1/F(p)$, particularly when the system transfer function contains zeros, as the bandwidth of $1/A^*(p)$ may be much smaller than the bandwidth of the system.

Therefore, the recommended choice of the denominator polynomial of SVF is to select $F(p)$ to be as close to $A^*(p)$ as possible. This is different to the common suggestion given in the existing literature (see e.g. [1, p. 102]), which recommends the bandwidth of $1/F(p)$ to be chosen closer to the bandwidth of the system.

4.4 Some Advantages and Trade-offs concerning the BEIVSVF Estimator

One of the differences between the proposed BEIVSVF estimator and the SRIVC estimator [23] is that the instrument and regressor vectors in the BEIVSVF estimator are not updated iteratively. This means that the inverse of the modified normal matrix in the BEIVSVF algorithm only needs to be computed once instead of being repeatedly updated as in the SRIVC algorithm. While preserving the consistency property, the BEIVSVF estimator is therefore more computationally efficient than the SRIVC estimator.

The BEIVSVF estimator does not achieve a relatively faster computation speed without any costs. A consequence of the one-off computation of the modified normal matrix is that the BEIVSVF is not statistically efficient under the output error model structure since its covariance matrix does not achieve the Crámer-Rao lower bound in [10, Theorem 1]. In addition, as seen from the expressions of (36) and (37) in Statement 3 of Theorem 2, the eigenvalues of $\mathbf{I} - \mathbf{Q}^{-1}\mathbf{R}$ not only depend on the choice of the auxiliary model $B_{ls}(p)/A_{ls}(p)$, the SVF filter $1/F(p)$ and the input signal $u(t)$ but also on the assumed intersample behaviour when discretising the filter $\bar{A}(p)/F(p)$. Hence, all these factors may affect the local convergence of the BEIVSVF estimator.

When comparing Statement 3 of Theorem 2 to the convergence result of the SRIVC estimator [8, Theorem 1, Statement 3], it can be seen that the SRIVC estimator has a stronger convergence guarantee since it has been shown [8] that

$$\boldsymbol{\theta}_{j+1}^{SRIVC} = \bar{\boldsymbol{\theta}} + o_p(\|\boldsymbol{\theta}_j^{SRIVC} - \bar{\boldsymbol{\theta}}\|),$$

that is, in the asymptotic case, the SRIVC algorithm locally converges in one iteration. In the case of the BEIVSVF estimator, we have traded off this guarantee for local convergence with the simplicity of the algorithm, i.e. the prefilters are fixed instead of being updated iteratively inside the algorithm, which is the condition that ensures the stronger convergence result. In terms of the convergence rate, the SRIVC estimator can be formulated as a Gauss-Newton type method (see e.g. [21, Section 5.2]) with a non-symmetric Hessian matrix due to the use of the instrument vector; however, the Hessian matrix is very close to being symmetric near the converging point for large sample sizes and small sampling periods. The large sample sizes ensure that the parameters of the prefilter in the instrument vector converges to the true parameters, whereas the small sampling periods ensure that the interpolation error terms that arise from constructing the filtered output during the SRIVC iterations are near zero. Hence, the SRIVC estimator approximately possesses a quadratic convergence rate. On the other hand, the convergence for the BEIVSVF estimator is linear due to the non-zero $\mathbf{I}-\mathbf{Q}^{-1}\mathbf{R}$ term in (42), which gives a slower convergence rate when high accuracy is desired [14].

5 Simulations

In this section, Monte Carlo (MC) simulations are performed to provide empirical evidence to support the theoretical results developed in Sections 3 and 4. The system is chosen to be

$$G(p) = \frac{1}{0.04p^2 + 0.2p + 1},$$

where the true parameters are given by

$$\boldsymbol{\theta}^* = \begin{bmatrix} 0.04 & 0.2 & 1 \end{bmatrix}^\top .$$

The MC simulations are performed with two different sampling periods. The signals in the first instance are sampled at $T = 0.1$ s, which is equivalent to a sampling frequency that is approximately 10 times the system bandwidth, whereas those in the second instance are sampled at $T = 0.02$ s. The system input is chosen to be a ZOH multisine with angular frequencies $0.5, 2, 5$ and 7 rad/s. The additive noise on the output is an i.i.d. Gaussian sequence with a zero mean and a variance of 0.1. The consistency property of the IVSVF and

BEIVSVF estimators is investigated by examining the mean and variance of the estimates in an MC study as the sample size, N, increases. The SRIVC estimator is employed as a benchmark to the simulation results as it has been proven to be consistent [8] and asymptotically efficient [10] under the output error model structure. The sample size, N, is varied from 500 to 10^6 in a logarithmic scale, where a total of 50 different sample sizes are used. Three hundred MC simulations are performed for each value of N. The maximum iteration is set to 100 and the tolerance ϵ set to 10^{-14} for the BEIVSVF and SRIVC algorithms. The same termination condition is used for the BEIVSVF and SRIVC algorithms as defined in (35). The mean and variance for the two sampling periods as a function of the number of samples are shown in Figures 1 to 5.

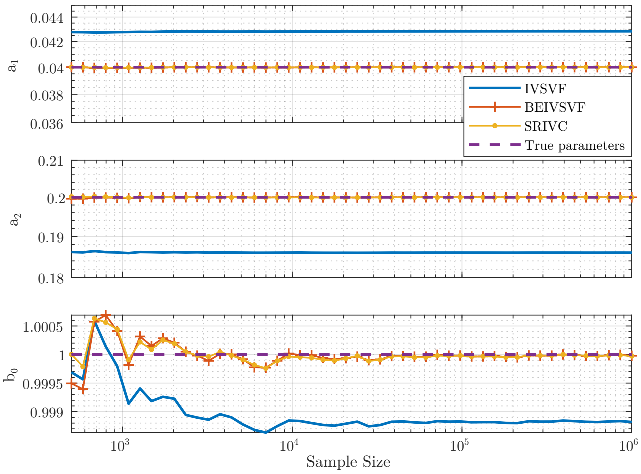

Fig. 1. Mean of the estimated parameters ($T = 0.1$ s).

For the case of $T = 0.1$ s, it can be seen in Figure 1 that the mean of the IVSVF estimates does not converge to the true parameters, which shows empirically that the estimator is not consistent. On the other hand, a zoomed-in version of Figure 1 shows that the mean values of both the BEIVSVF and SRIVC estimates converge to the true parameters in Figure 2. Given that the variances of the estimates in Figure 3 are decreasing with an increasing sample size as well as the fact that the consistency property of the SRIVC estimator has already been confirmed previously [8], this then provides empirical evidence to the consistency of the BEIVSVF estimator.

Similar results can be observed in Figures 4 and 5, where the sampling period is chosen to be $T = 0.02$ s, which is equivalent to a sampling frequency that is 50 times faster than the system bandwidth. Although the mean of the IVSVF estimates are much closer to the true parameters when fast sampling is used, a discrepancy between the mean of the IVSVF estimates and the true parameters can still be observed in Figure 4. It should be noted that fast sampling may not be available in all practical situations due to equipment limitations and/or the nature of

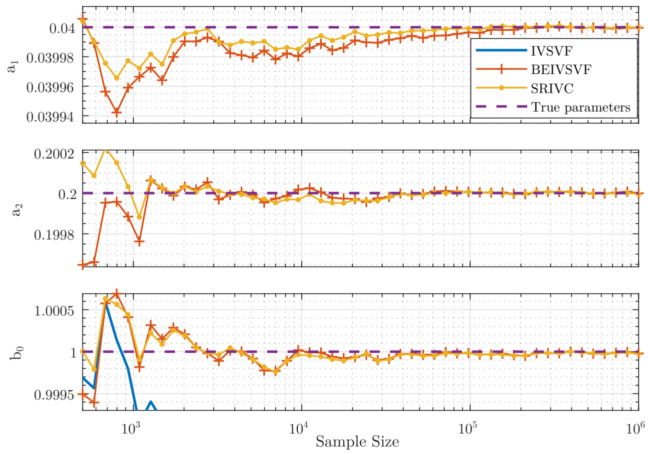

Fig. 2. Zoomed-in version of the mean of the estimated parameters ($T = 0.1$ s).

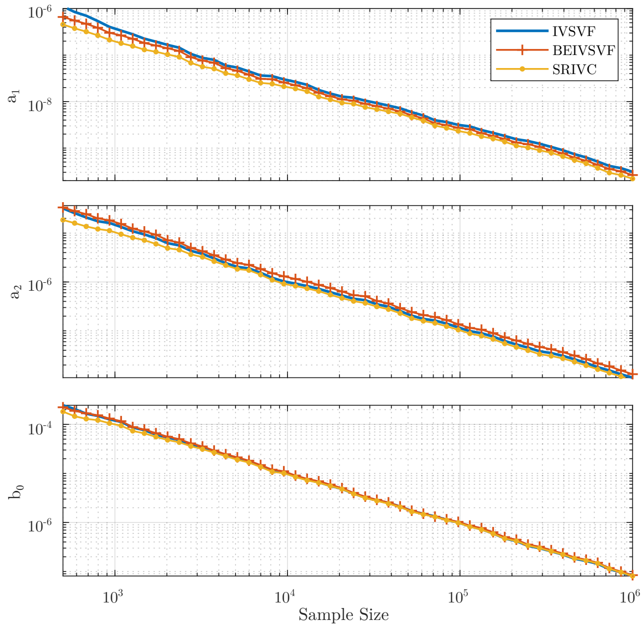

Fig. 3. Variance of the estimated parameters ($T = 0.1$ s).

the systems being identified. Hence, the sampling period should not be relied upon to reduce the bias in the estimates. The BEIVSVF and SRIVC estimators, on the other hand, are still converging to the true parameters with a decreasing variance.

The computation times of the BEIVSVF and SRIVC estimators for the 15000 MC simulations (300 MC for 50 different sample sizes) for the two sampling periods are displayed in Table 1. The simulations are performed on a computer with an Intel Xenon 3.5GHz processor, and the `srivc` function from the CONTSID toolbox is used. It can be seen that the BEIVSVF algorithm is 2 to 3 times faster than the SRIVC algorithm under the same simulations conditions while being able to achieve similar statistical results.

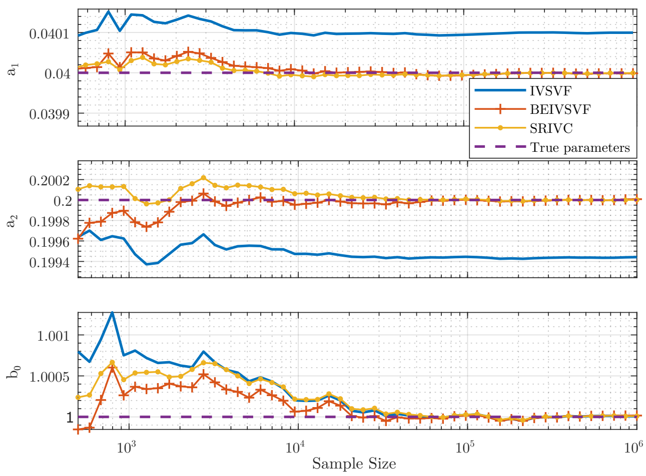

Fig. 4. Mean of the estimated parameters ($T = 0.02$ s).

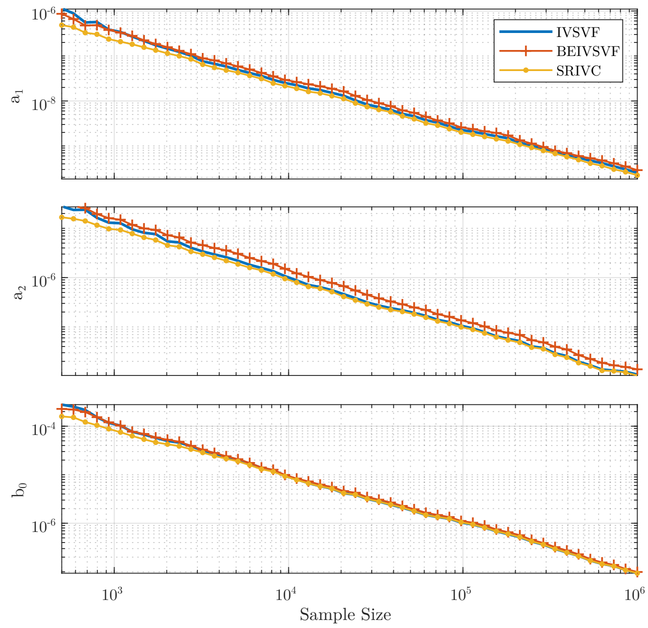

Fig. 5. Variance of the estimated parameters ($T = 0.02$ s).

Table 1
Simulation time of the BEIVSVF and SRIVC algorithms for a total of 15000 MC runs.

Sampling periods	$T = 0.1$	$T = 0.02$
BEIVSVF	7.62 hr	3.78 hr
SRIVC	15.91 hr	10.94 hr

6 Conclusion

In this paper, we have analysed the consistency of the IVSVF estimator by taking into account the intersample behaviour of the input and output signals. It has been found that the IVSVF estimator is not consistent when only sampled input and output signals are available, even if the input intersample behaviour is exactly known. This is due to the interpolation error that arises from constructing the filtered output. The BEIVSVF estimator has been proposed to remove the asymptotic bias caused by this interpolation error and its consistency property has been proven. The use of the proposed estimator has also been discussed from a practical standpoint. Monte Carlo simulations have been performed to verify the consistency of the BEIVSVF estimator.

7 Acknowledgements

This work was in part supported by the Australian government Research Training Program (RTP) scholarship, and in part by the Swedish Research Council under contract number 2016-06079 (NewLEADS).

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